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Gorenstein Algebras and Hochschild Cohomology

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To Mel Hochster on his 65th birthday

Introduction

Each one of the main classes of commutative Noetherian rings—regular, complete intersection, Gorenstein, and Cohen–Macaulay—is defined by *local* properties that require verification at *every* maximal ideal. It is therefore important to develop for these properties *global* recognition criteria involving only *finitely many* invariants. Finitely generated algebras over fields provide the test case. Our goal is to devise finitistic global tests applicable also in a more general, relative situation.

To fix notation, let K be a commutative Noetherian ring and $\sigma\colon K \to S$ a *flat* homomorphism of rings, which is *essentially of finite type*; σ is said to be Cohen–Macaulay or Gorenstein if its fiber rings have the corresponding property (see Section 2 for details). The following result is taken from Theorem 4.2; recall that $\operatorname{grade}_P S$ is the smallest integer n with $\operatorname{Ext}^n_P(S,P) \neq 0$.

Theorem 1. *Assume that* $\operatorname{Spec} S$ *is connected. Let* $K \to P \to S$ *be a factorization of* σ *with* P *a localization of a polynomial ring* $K[x_1,\dots,x_d]$ *and* S *a finite* P*-module.*

The map σ *is Cohen–Macaulay if for* $g = \operatorname{grade}_P S$ *one has*

$$\operatorname{Ext}^n_P(S,P) = 0 \quad \textit{for } g < n \le g + d.$$

Conversely, if σ *is Cohen–Macaulay, then* $\operatorname{Ext}^n_P(S,P) = 0$ *holds for* $n \neq g$.

The homomorphism σ *is Gorenstein if and only if it is Cohen–Macaulay and the* S*-module* $\operatorname{Ext}^g_P(S,P)$ *is invertible.*

Thus, it is easy to recognize Cohen–Macaulay maps, because they are characterized in terms of *vanishing* of cohomology in a *finite* number of *specified* degrees; this can be decided from finite constructions. On the other hand, the condition needed to identify the Gorenstein property involves the *structure* of $\operatorname{Ext}^g_P(S,P)$ as a module over S, which is not determined by finitistic data (see Remark 5.4).

Partly motivated by recent characterizations of regular homomorphisms (recalled in Section 5), we approach the problem by studying the homological properties of S as a module over the *enveloping algebra* $S^{\mathrm{e}} = S \otimes_K S$, which acts on

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