Michigan Math. J. 57 (2008)

Lower Bounds for Hilbert–Kunz Multiplicities in Local Rings of Fixed Dimension

Ian M. Aberbach & Florian Enescu

Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a local ring of positive characteristic p—that is, quasi-local (only one maximal ideal) and Noetherian. Let $q = p^e$, where e is a nonnegative integer. For any ideal I of R we denote $I^{[q]} = (i^q : i \in I)$.

For an $\mathfrak{m}$-primary ideal I, one can consider the Hilbert–Samuel multiplicity and the Hilbert–Kunz multiplicity of I with respect to R.

Definition 1.1. Let I be an $\mathfrak{m}$-primary ideal in $(R, \mathfrak{m})$. Let $\lambda(\cdot)$ denote the usual length function.

1. *The Hilbert–Samuel multiplicity of R at I* is defined by

$$\mathrm{e}(I) = \mathrm{e}(I; R) := \lim_{n\to\infty} d!\,\frac{\lambda(R/I^n)}{n^d}.$$

The limit exists and is a positive integer.

2. *The Hilbert–Kunz multiplicity of R at I* is defined by

$$\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}_{\mathrm{HK}}(I; R) := \lim_{q\to\infty} \frac{\lambda(R/I^{[q]})}{q^d}.$$

Monsky has shown that the latter limit exists and is positive.

The Hilbert–Samuel multiplicty of R, denoted $\mathrm{e}(R)$, is by definition $\mathrm{e}(\mathfrak{m})$. Similarly, the Hilbert–Kunz multiplicity of R, denoted $\mathrm{e}_{\mathrm{HK}}(R)$, is $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$.

It is known that, for parameter ideals I, one has $\mathrm{e}(I) = \mathrm{e}_{\mathrm{HK}}(I)$. The following sequence of inequalities is also known to hold whenever I is $\mathfrak{m}$-primary:

$$\max\left\{1, \frac{\mathrm{e}(I)}{d!}\right\} \le \mathrm{e}_{\mathrm{HK}}(I) \le \mathrm{e}(I).$$

We call a local ring R *formally unmixed* if $\hat{R}$ is equidimensional and $\mathrm{Min}(\hat{R}) = \mathrm{Ass}(\hat{R})$—in other words, $\dim(\hat{R}/P) = \dim(\hat{R})$ for all its minimal primes P and if all associated primes of $\hat{R}$ are minimal. Nagata calls such rings *unmixed.* However, throughout our paper, a local unmixed ring is a local ring R that is equidimensional and for which $\mathrm{Min}(R) = \mathrm{Ass}(R)$.

Received June 29, 2007. Revision received November 14, 2007.

The second author was partially supported by a Young Investigator Grant H98230-07-1-0034 from the National Security Agency.