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Conway Products and Links with Multiple Bridge Surfaces

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1. Introduction

A link K in a 3-manifold M is said to be in *bridge position* with respect to a Heegaard surface P for M if each arc of $K - P$ is parallel to P, in which case P is called a *bridge surface* for K in M. Given a link in bridge position with respect to P, it is easy to construct more complex bridge surfaces for K from P—for example, by stabilizing the Heegaard surface P or by perturbing K to introduce a minimum and an adjacent maximum. As with Heegaard splitting surfaces for a manifold, it is likely that most links have multiple bridge surfaces even apart from these simple operations. In an effort to understand how two bridge surfaces for the same link might compare, it seems reasonable to follow the program used in [RS] to compare distinct Heegaard splittings of the same non-Haken 3-manifold. The restriction to non-Haken manifolds ensured that the relevant Heegaard splittings were strongly irreducible. In our context the analogous condition is that the bridge surfaces are c-weakly incompressible (definition to follow). The natural analogy to the first step in [RS] would be to demonstrate that any two distinct c-weakly incompressible bridge surfaces for a link K in a closed orientable 3-manifold M can be isotoped so that their intersection consists of a nonempty collection of curves, each of which is essential (including nonmeridional) on both surfaces. In some sense the similar result in [RS] could then be thought of as the special case in which $K = \emptyset$.

Here we demonstrate that this is true when there are no incompressible Conway spheres for the knot K in M (cf. Section 4 and [GL]). In the presence of Conway spheres a slightly different outcome cannot be ruled out: the bridge surfaces each intersect a collar of a Conway sphere in a precise way; outside the collar the bridge surfaces intersect only in curves that are essential on both surfaces; and inside the collar there is inevitably a single circle intersection that is essential in one surface and meridional—and hence inessential—in the other.

2. Definitions and Notation

Suppose that K is a properly embedded collection of 1-manifolds in a compact manifold M. For X any subset of M, let X_K denote $X - K$. A disk that meets K

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