Michigan Math. J. 56 (2008)

Klein's Conjecture for Contact Automorphisms of the Three-Dimensional Affine Space

Marat Gizatullin

1. Introduction

The ground field k is of characteristic 0.

Let (x, y, p) be three affine coordinates. The Pfaffian form

$$\omega = dy - p\,dx \tag{1.1}$$

is said to be a *contact form* of the three-dimensional space. A birational transformation T of the three-dimensional (x, y, p)-space defined by

$$x' = f(x, y, p), \quad y' = g(x, y, p), \quad p' = h(x, y, p) \tag{1.2}$$

is said to be a *contact Cremona transformation* of the (x, y)-plane if the transform $T^*(\omega)$ of the contact form (1.1) is proportional to this form:

$$T^*(\omega) = \rho(x, y, p) \cdot \omega, \tag{1.3}$$

where $\rho(x, y, p)$ is a nonzero rational function. Our reference to a "plane" may seem mistaken, but classically it means an action of the transformation on plane contact elements $((x, y), p)$ consisting of a point (x, y) and a slope p at the point. We will say that $\rho(x, y, p)$ is the *multiplier* of T. The contact transformation T is said to be a *contact affine transformation* (or *contact polynomial automorphism*) if T and its inverse T^{-1} are polynomial. For a contact affine transformation of $\mathbb{A}^3$, the multiplier is a nonzero constant (see Lemma 2.1 for contact transformations of odd-dimensional spaces).

Example 1.1. Let

$$x' = f(x, y), \qquad y' = g(x, y) \tag{1.4}$$

be a Cremona transformation of the (x, y)-plane. It is possible to extend (1.4) to a contact transformation

$$x' = f(x, y), \quad y' = g(x, y), \quad p' = h(x, y, p), \tag{1.5}$$

where

$$h(x, y, p) = \frac{p\dfrac{\partial g}{\partial y} + \dfrac{\partial g}{\partial x}}{p\dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial x}}.$$

Received November 16, 2006. Revision received February 26, 2007.