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A Singular-Hyperbolic Closing Lemma

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1. Introduction

The fundamental problem in the qualitative theory of dynamical systems is the study of the asymptotic behavior of the orbits in a given system. This problem gave rise to the concept of an *attracting set,* which (roughly speaking) is a place where a large set of positive orbits of the system go. On the other hand, the dynamical classification of the attracting sets is an extremely difficult problem that often requires extra structures. One such structure is *Smale's hyperbolicity,* which consists of a tangent bundle decomposition formed by a contracting and an expanding subbundle together with the flow's direction. There is now a rich theory of *hyperbolic attractors* (i.e., hyperbolic transitive attracting sets) dealing with the dynamical, geometric, ergodic, and topological properties of these objects. Nevertheless, this theory does not include such important examples as the *Henón attractor* [BeC], where there is a positive Lyapunov exponent at dense orbits; the *geometric Lorenz attractor* [AfByS; GW], where there is a critical point accumulated by periodic orbits; or even the *singular horseshoe* [LPa], which is not attracting but has properties resembling the Lorenz attractor. For the Lorenz attractor, R. Bowen, Y. Pesin, and Y. Sinai observed early on that it displays interesting properties (such as as robust transitivity and denseness of periodic orbits) that are usually associated with hyperbolic attractors. The explanation of these properties goes back to end of the nineties, when [ST] introduced what they called a *wild attractor*—that is, an example of an attractor in dimension 4 that simultaneously exhibits spiraling singularities and persistent homoclinic tangencies. The wild attractor has nothing to do with the hyperbolic properties of the Lorenz attractor, but it was noticed that the former displays a partially hyperbolic splitting *with volume-expanding central subbundle*. This subtle property was then used in [MPaP1] to define *singular-hyperbolic set* as a partially hyperbolic set with hyperbolic singularities and volume-expanding central subbundle. With this definition in hand, [MPaP1] proved that a C^1 robust transitive set with singularities is a singular-hyperbolic attractor for either the flow or the reverse flow (see also [MPaP3]). This result motivates the study of the dynamical properties of singular-hyperbolic sets taking the hyperbolic dynamical systems as a model.

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