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# Toledo Invariants of 2-Orbifolds and Higgs Bundles on Elliptic Surfaces

Mike Krebs

## Introduction

In this paper we investigate the space of conjugacy classes of semisimple representations $\rho\colon \pi_1^{\mathrm{orb}}(O) \to U(2,1)$ for 2-orbifolds arising as the base of a Seifert fibration. To each connected component in the corresponding representation variety, we associate a number called the *orbifold Toledo invariant*. Our main result (Theorem 6.2) explicitly computes all values that the orbifold Toledo invariant takes on when the Seifert manifold $Y$ is a homology 3-sphere. One thereby obtains (Corollary 6.3(a)) a lower bound for the number of connected components in the representation variety. Our results also yield (Corollary 6.3(b)) a lower bound for the number of connected components in the space of conjugacy classes of irreducible representations $\rho\colon \pi_1(Y) \to \mathrm{PU}(2,1)$.

In [38], Toledo introduces an invariant $\tau$ for representations of the fundamental group of an oriented 2-manifold $M$ into $\mathrm{PU}(p,1)$. This invariant can be viewed as a map $\tau\colon \mathrm{Hom}(\pi_1(M), \mathrm{PU}(p,1)) \to \mathbb{R}$. As discussed in Section 1, the construction of the Toledo invariant is quite general: one may replace $M$ by an arbitrary topological space and $\mathrm{PU}(p,1)$ by any topological group $G$. We shall be concerned with a compact Kähler manifold $M$ and a group $G$ of the form $\mathrm{PU}(p,q)$; under these circumstances, representations that take on distinct Toledo invariants necessarily lie in distinct components of the corresponding representation space.

In order to discuss some previous results on Toledo invariants, we now introduce some notation that will be used throughout the paper. If $\pi$ is any group and $G$ is a Lie group with Lie algebra $\mathfrak{g}$, then we shall say that a representation $\rho\colon \pi \to G$ is *irreducible* (resp. *semisimple*) if the action of $\pi$ on $\mathfrak{g}$ induced via $\mathrm{ad}(\rho)$ is irreducible (resp. semisimple). We denote the set of irreducible representations $\rho\colon \pi \to G$ by $\mathrm{Hom}^*(\pi, G)$ and the set of semisimple representations by $\mathrm{Hom}^+(\pi, G)$. Endow $\mathrm{Hom}(\pi, G)$ with the point-open topology, and regard $\mathrm{Hom}^*(\pi, G)$ and $\mathrm{Hom}^+(\pi, G)$ as subspaces. (Note that if $\pi$ is finitely generated with generators $t_1, \dots, t_n$, then $\mathrm{Hom}(\pi, G)$ is homeomorphic to the closed subspace $\{(x_1, \dots, x_n) \in G^n \mid r_\alpha(x_1, \dots, x_n) = 1\}$ of $G^n$, where the $r_\alpha$ range over all relations between the $t$s.) Let $G$ act on $\mathrm{Hom}(\pi, G)$ by conjugation. For any space $M$, let $\mathcal{R}_G(M)$, $\mathcal{R}^*_G(M)$, and $\mathcal{R}^+_G(M)$ denote the quotients by this action of $\mathrm{Hom}(\pi_1(M), G)$, $\mathrm{Hom}^*(\pi_1(M), G)$, and $\mathrm{Hom}^+(\pi_1(M), G)$, respectively.

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