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Invariant Differential Operators Associated with a Conformal Metric

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1. Introduction

Peschl defined invariant higher-order derivatives of a holomorphic or meromorphic function on the unit disk. Here, the invariance is concerned with the hyperbolic metric of the source domain and the canonical metric of the target domain. Minda and Schippers extended Peschl's invariant derivatives to the case of general conformal metrics. We introduce similar invariant derivatives for smooth functions on a Riemann surface and show a complete analogue of Faà di Bruno's formula for the composition of a smooth function with a holomorphic map with respect to the derivatives. An interpretation of these derivatives in terms of intrinsic geometry and some applications will be also given.

The uniformization theory tells us that an arbitrary Riemann surface has the natural geometry—namely, spherical, Euclidean, or hyperbolic geometry. Standard examples are the Riemann sphere $\hat{\mathbb{C}}$ with the spherical metric $|dz|/(1+|z|^2)$, the complex plane with the Euclidean metric $|dz|$, and the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the hyperbolic (or the Poincaré) metric $|dz|/(1-|z|^2)$. For a unifying treatment, we introduce the notation $\mathbb{C}_\varepsilon$ to designate $\hat{\mathbb{C}}$ for $\varepsilon = 1$, $\mathbb{C}$ for $\varepsilon = 0$, and $\mathbb{D}$ for $\varepsilon = -1$. Unless otherwise stated, we understand that $\mathbb{C}_\varepsilon$ is equipped with the canonical metric $\lambda_\varepsilon(z)|dz| = |dz|/(1+\varepsilon|z|^2)$. Note that λ_ε has constant Gaussian curvature 4ε.

For a holomorphic map $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ $(\delta, \varepsilon = 1, 0, -1)$, it is more natural to consider a type of invariant derivatives of $f(z)$ associated with $\mathbb{C}_\delta$ and $\mathbb{C}_\varepsilon$ rather than the usual derivatives $f^{(n)}(z) = d^n f(z)/dz^n$. As such, commonly used is the invariant derivative $D^n f(z)$ due to Peschl [Pe], which is defined by the power series expansion

$$\frac{f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right) - f(z)}{1+\varepsilon\overline{f(z)}\,f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right)} = \sum_{n=1}^{\infty} \frac{D^n f(z)}{n!} \cdot \zeta^n \tag{1.1}$$

around $\zeta = 0$. Note that the group $\mathrm{Isom}^+(\mathbb{C}_\varepsilon)$ of sense-preserving isometries of $\mathbb{C}_\varepsilon$ consists of the maps $L(\zeta) = \eta(\zeta - a)/(1+\varepsilon\bar{a}\zeta)$ for some $a \in \mathbb{C}_\varepsilon$ and $\eta \in \mathbb{C}$ with $|\eta| = 1$, where $L(\zeta) = -\eta/\zeta$ for $\varepsilon = 1$ and $a = \infty$. For example,

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