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Monads and Regularity of Vector Bundles on Projective Varieties

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1. Introduction

In the seventies, Horrocks showed that every vector bundle E on $\mathbb{P}^2$ and $\mathbb{P}^3$ admits a certain "double-ended resolution" by line bundles that he called *monads.* Monads appear in a wide variety of contexts within algebraic geometry, and they are now one of the most important tools that permit us to construct vector bundles with prescribed invariants (e.g., rank, determinant, Chern classes) and to study them with the methods of linear algebra. A large number of vector bundles on $\mathbb{P}^n$ are the cohomology of monads, and most of them can be constructed as cohomologies of quasi-linear monads (see Definition 2.9). Indeed, any vector bundle on $\mathbb{P}^3$ is the cohomology of a quasi-linear monad, and any instanton bundle on $\mathbb{P}^{2l+1}$ is also the cohomology of a quasi-linear monad. On the other hand, in the mid-sixties Mumford introduced the concept of regularity for a coherent sheaf F on a projective space $\mathbb{P}^n$. Since then, Castelnuovo–Mumford regularity has become a fundamental invariant—in both commutative algebra and algebraic geometry—for measuring the complexity of a sheaf. In [4; 5] we generalized the notion of Castelnuovo–Mumford regularity for coherent sheaves on projective spaces and also introduced a notion of regularity for coherent sheaves on other projective varieties (as multiprojective spaces, Grassmanians, and hyperquadrics $Q_n \subset \mathbb{P}^{n+1}$) that is equivalent to the Castelnuovo–Mumford regularity when the variety is a projective space $\mathbb{P}^n$.

The main goal of this paper is to bound the regularity of vector bundles E on $\mathbb{P}^n$ and Q_n defined as the cohomology of quasi-linear monads. As a by-product, we obtain a bound for the regularity of any vector bundle on $\mathbb{P}^3$, the regularity of any instanton bundle on $\mathbb{P}^{2l+1}$, and the regularity of any instanton bundle on a hyperquadric $Q_{2l+1} \subset \mathbb{P}^{2l+2}$.

The paper is organized as follows. In Section 2, we briefly recall the notion of regularity for coherent sheaves on projective varieties and summarize its main formal properties. We also recall the use of monads for constructing vector bundles and introduce the notion of a quasi-linear monad. Sections 3 and 4 are the heart of the paper. In Section 3, we give effective bounds for the regularity of any coherent sheaf on $\mathbb{P}^n$ that is the cohomology of a quasi-linear monad (see Theorem 3.2). In particular, we bound the regularity of any vector bundle on $\mathbb{P}^3$ as well as the

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