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On Homaloidal Polynomials

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Let $\mathbb{P}^n$ be the projective space over a field k. If F is a homogeneous polynomial, we say that F is *homaloidal* if the polar map ∂F defined by the partial derivatives of F is a birational selfmap of $\mathbb{P}^n$. Although the problem of determining homaloidal polynomials has a classical flavor, the theme was only recently raised in an algebro-geometric context by Dolgachev [Do] following suggestions stemming from the theory of prehomogeneous varieties: the relative invariants of prehomogeneous spaces are, in fact, homaloidal polynomials [EKP; KiSa]. Dolgachev classifies square free homaloidal polynomials in $\mathbb{P}^2$ (see also [D]) and characterizes square free homaloidal polynomials in $\mathbb{P}^3$ that are products of four independent linear forms. Dolgachev also raises the following question: Is it true that a non–square free product of linear forms is homaloidal if and only if the product of its factors with multiplicity 1 is? This question has been given a positive answer in a specific case (see [KrS]) and in full generality (see [DP]) in a topological context. We will give an algebraic proof of the following result.

Theorem A. *Suppose that k is of characteristic* 0. *Let $L_0, \dots, L_r$ be linear forms and let $m_0, \dots, m_r$ be positive integers. Then $F = \prod_{i=0}^r L_i^{m_i}$ is a homaloidal polynomial if and only if* (i) *$F_{\text{red}} = \prod_{i=0}^r L_i$ is homaloidal and* (ii) *$r = n$ and the linear forms $L_0, \dots, L_n$ are independent.* (Here the subscript "red" denotes "reduced".)

In particular, square free homaloidal polynomials that split as the product of linear forms all induce, up to a projectivity, standard Cremona transformations. The hypothesis on the characteristic of the ground field is essential because we need resolution of singularities. When this is possible, we obtain the following result.

Theorem B. *Assume that resolution of singularities holds in characteristic p and dimension n (e.g., if $n = 2$ or $n = 3$ and $p \geq 7$). Then $F = \prod_{i=0}^r L_i^{m_i}$ is a homaloidal polynomial if and only if* (i) *p does not divide m_i for each i and* (ii) *$F_{\text{red}} = \prod_{i=0}^r L_i$ is homaloidal if and only if p does not divide m_i for each i and $r = n$ and the linear forms $L_0, \dots, L_n$ are independent.*

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