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The Zeros of Flat Gaussian Random Holomorphic Functions on $\mathbb{C}^n$, and Hole Probability

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1. Introduction

Random polynomials and random holomorphic functions are studied as a way of gaining insight into difficult problems such as string theory and analytic number theory. A particularly interesting case of random holomorphic functions is when the functions can be defined so that they are invariant with respect to the natural isometries of the space in question. The class of functions that we will study are the unique Gaussian random holomorphic functions, up to multiplication by a nonzero holomorphic function, whose expected zero set is uniformly distributed on $\mathbb{C}^n$. Such functions are also known as the flat Gaussian random holomorphic functions. For a random holomorphic function of this class, we will determine the expected value of the unintegrated counting function for a ball of large radius as well as the chance that there are no zeros present—a pathological event that is known as a "hole". In so doing we generalize a result of Sodin and Tsirelison to n dimensions in order to give the first nontrivial example where the hole probability is computed in more than one complex variable.

The topic of random holomorphic functions is an old one, with many results from the first half of the twentieth century, that is recently experiencing a renaissance. In particular, Kac determined a formula for the expected distribution of zeros of real polynomials in a certain case [5], and this work was subsequently generalized throughout the years [3]. A series of papers by Offord [7; 8] is particularly relevant to questions involving the hole probability of random holomorphic functions and the distribution of values of random holomorphic functions. There has been a flurry of recent interest in the zero sets of random polynomials and holomorphic functions, which are much more natural objects than they may initially appear. For example, Bleher, Shiffman, and Zelditch [1] show that, for any positive line bundle over a compact complex manifold, the random holomorphic sections to L^N (defined intrinsically) have universal high N correlation functions.

In addition to results describing the typical behavior, there have also been several results in one (real or complex) dimension for Gaussian random holomorphic functions where the hole probability has been determined. For a specific class of

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