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Fixed Points and Determining Sets for Holomorphic Self-Maps of a Hyperbolic Manifold

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0. Introduction

Let M be a complex manifold. Then $H(M, M)$ is the set of holomorphic maps from M to M, that is, the set of endomorphisms of M. A special case of endomorphisms are automorphisms of M, $\operatorname{Aut}(M) \subset H(M, M)$.

Definition 0.1. A set $K \subset M$ is called a *determining subset* of M with respect to $\operatorname{Aut}(D)$ ($H(M, M)$ resp.) if, whenever g is an automorphism (endomorphism resp.) such that $g(k) = k$ for all $k \in K$, then g is the identity map of M.

The notion of a determining set was first introduced in a paper written by the first two authors in collaboration with Steven G. Krantz and Kang-Tae Kim [F+1]. That paper was an attempt to find a higher-dimensional analog of the following result of classical function theory [PL]: If $f : M \to M$ is a conformal self-mapping of a plane domain M that fixes three distinct points, then $f(\zeta) = \zeta$.

This one-dimensional result is true even for endomorphisms of a bounded domain $D \subset\subset \mathbb{C}$. To prove this, one may first use the well-known theorem stating that if an endomorphism of D fixes two distinct points then it is an automorphism; then use the [PL] theorem. In the n-dimensional case, determining sets (for automorphisms and endomorphisms) of bounded domains in $\mathbb{C}^n$ have been investigated in [F+2; FMa; KiKr; V1; V2].

Let $W_s(M)$ denote the set of s-tuples $(x_1, \dots, x_s)$, where $x_j \in M$, such that $\{x_1, \dots, x_s\}$ is a determining set with respect to $\operatorname{Aut}(M)$. Similarly, $\hat{W}_s(M)$ denotes the set of s-tuples $(x_1, \dots, x_s)$ such that $\{x_1, \dots, x_s\}$ is a determining set with respect to $H(M, M)$. Hence $\hat{W}_s(M) \subseteq W_s(M) \subseteq M^s$. We now introduce two values $s_0(M)$ and $\hat{s}_0(M)$. If $\operatorname{Aut}(M) = \mathrm{id}$ then $s_0(M) = 0$; otherwise, $s_0(M)$ is the least integer s such that $W_s(M) \neq \emptyset$. If $W_s(M) = \emptyset$ for all s then $s_0(M) = \infty$. Analogously, $\hat{s}_0(M)$ denotes the least integer s such that $\hat{W}_s(M) \neq \emptyset$; if no such integer exists (i.e., if $\hat{W}_s(M) = \emptyset$ for all s) then $\hat{s}_0(M) = \infty$. In all cases $s_0(M) \le \hat{s}_0(M)$.

The main objectives of this paper are (1) to generalize the results for bounded domains in $\mathbb{C}^n$ to hyperbolic manifolds and (2) to illustrate that the results are quite different for the nonhyperbolic manifolds.

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