

Changes of Variables in ELSV-type Formulas

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1. Introduction

In [5], Goulden, Jackson, and Vakil formulated a conjecture relating certain Hurwitz numbers (enumerating ramified coverings of the sphere) to the intersection theory on a conjectural Picard variety $\text{Pic}_{g,n}$. This variety, of complex dimension $4g - 3 + n$, is supposedly endowed with a natural morphism to the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. The fiber over a point $x \in \mathcal{M}_{g,n}$ lying in the open part of the moduli space is equal to the Jacobian of the corresponding smooth curve C_x . The variety $\text{Pic}_{g,n}$ is also supposed to carry a universal curve $\mathcal{C}_{g,n}$ with n disjoint sections s_1, \dots, s_n . Denote by \mathcal{L}_i the pull-back under s_i of the cotangent line bundle to the fiber of $\mathcal{C}_{g,n}$. Then we obtain n tautological 2-cohomology classes $\psi_i = c_1(\mathcal{L}_i)$ on $\text{Pic}_{g,n}$.

We shall use the formula from [5] to study the intersection numbers of the classes ψ_i on $\text{Pic}_{g,n}$ (if it is ever to be constructed). In particular, we prove a Witten–Kontsevich-type theorem relating the intersection theory and integrable hierarchies. These equations, together with the string and dilation equations, allow us to compute all the intersection numbers under consideration.

Independently of the conjecture of [5], our results can be interpreted as meaningful statements about Hurwitz numbers. Our methods are close to those of Kazarian and Lando in [7] and make use of Hurwitz numbers. We also extend the results of [7] to include the Hodge integrals over the moduli spaces involving one λ -class.

1.1. The Conjecture

Fix n positive integers b_1, \dots, b_n . Let $d = \sum b_i$ be their sum.

DEFINITION 1.1. The number of degree- d ramified coverings of the sphere by a genus- g surface possessing

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