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A Dirichlet Problem for the Complex Monge–Ampère Operator in $\mathcal{F}(f)$

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Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a hyperconvex domain: a connected open set that admits a negative plurisubharmonic exhaustion function. Throughout this paper it is always assumed that Ω is bounded. The class of plurisubharmonic functions defined on Ω will be denoted $\mathcal{PSH}(\Omega)$. In the theory of distributions, the smooth functions with compact support—the so-called test functions—play an important role. Because there exist no plurisubharmonic functions with compact support in Ω that are not identically zero, it is useful to introduce $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$. This class has a role similar to that of the class of test functions, $C_0^\infty(\Omega)$, since $C_0^\infty(\Omega) \subset \mathcal{E}_0 \cap C(\bar{\Omega}) - \mathcal{E}_0 \cap C(\bar{\Omega})$ [9, Lemma 3.1]. A bounded plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ if $\lim_{z\to\xi} \varphi(z) = 0$ for every $\xi \in \partial\Omega$ and $\int_\Omega (dd^c\varphi)^n < +\infty$, where $(dd^c\cdot)^n$ is the complex Monge–Ampère operator. The maximum principle for plurisubharmonic functions implies that if $\varphi \in \mathcal{E}_0$ then $\varphi < 0$ or $\varphi = 0$. Bedford and Taylor proved in [4] that $(dd^c\cdot)^n$ is well-defined on $\mathcal{PSH}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. This implies that the definition of $\mathcal{E}_0$ is well-posed and that $(dd^c\cdot)^n$ is well-defined on $\mathcal{E}_0$.

Assume that u is a plurisubharmonic function defined on Ω and that $[\varphi_j]_{j=1}^\infty$, $\varphi_j \in \mathcal{E}_0$, is a decreasing sequence that converges pointwise to u on Ω as j tends to $+\infty$. If there can be no misinterpretation, a sequence $[\cdot]_{j=1}^\infty$ will be denoted by $[\cdot]$. For fixed $p \geq 1$, consider the following assertions:

(1) $\sup_j \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < +\infty$;
(2) $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$.

If the sequence $[\varphi_j]$ can be chosen such that (1) holds, then u is said to be in $\mathcal{E}_p$ $(= \mathcal{E}_p(\Omega))$; if (2) holds, then u is in $\mathcal{F}$ $(= \mathcal{F}(\Omega))$. Finally, if both (1) and (2) are satisfied then $u \in \mathcal{F}_p$ $(= \mathcal{F}_p(\Omega))$. In [9], Cegrell proved that the complex Monge–Ampère operator is well-defined on the subset $\mathcal{E}$ of nonpositive plurisubharmonic functions containing both $\mathcal{F}$ and $\mathcal{E}_p$ (see Section 1 or [9] for the definition of $\mathcal{E}$).

It is proved in Section 1 that, for $u \in \mathcal{F} \cup_{p\geq 1} \mathcal{E}_p$,

$$\limsup_{z\to\xi} u(z) = 0$$

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