Michigan Math. J. 55 (2007)

On the Maximum Principle and a Notion of Plurisubharmonicity for Abstract CR Manifolds

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0. Introduction

Let $\mathcal{M}$ be a smooth manifold and let $\mathcal{V}$ be a subbundle of $\mathbb{C}T\mathcal{M}$, the complexified tangent bundle of $\mathcal{M}$. The pair $(\mathcal{M},\mathcal{V})$ is called an *abstract CR manifold* if $\mathcal{V}$ is involutive and if, for each $p \in \mathcal{M}$, $\mathcal{V}_p \cap \bar{\mathcal{V}}_p = \{0\}$. Recall that $\mathcal{V}$ is involutive if the space of smooth sections of $\mathcal{V}$, $C^\infty(\mathcal{M},\mathcal{V})$, is closed under commutators. Let n be the complex dimension of the fibre $\mathcal{V}_p$ of $\mathcal{V}$ at p and write $\dim_{\mathbb{R}} \mathcal{M} = n+m$. The number n is called the *CR dimension* of $\mathcal{M}$, and $d = m-n$ will be called the *CR codimension* of $\mathcal{M}$. If $d = 1$, the CR structure is said to be of hypersurface type. The CR manifold $(\mathcal{M},\mathcal{V})$ is called *integrable* or *locally embeddable* if, for any $p_o \in \mathcal{M}$, there exist m complex-valued C^∞ functions $\mathcal{Z}_1,\dots,\mathcal{Z}_m$ defined near p_o such that (a) $L\mathcal{Z}_j = 0$ for all $L \in C^\infty(\mathcal{M},\mathcal{V})$, $j = 1,\dots,m$, and (b) the differentials $d\mathcal{Z}_1,\dots,d\mathcal{Z}_m$ are $\mathbb{C}$-linearly independent. Any such set of functions $\mathcal{Z}_j$ will be called a *complete set of first integrals.*

If $(\mathcal{M},\mathcal{V})$ is an integrable CR manifold, then the mapping $p \mapsto \mathcal{Z}(p) = (\mathcal{Z}_1(p),\dots,\mathcal{Z}_m(p)) \in \mathbb{C}^m$, where the $\mathcal{Z}_j$ are a complete set of first integrals, is a map of constant rank near p_o and so is an immersion. Thus, if U is a small neighborhood of p_o, then $\mathcal{Z}(U)$ is an embedded real submanifold of $\mathbb{C}^m$ of dimension $m+n$, and its real codimension in $\mathbb{C}^m$ agrees with the CR codimension $d = m-n$. It is easy to see that $\mathcal{Z}(U)$ is a generic CR submanifold of $\mathbb{C}^m$ and that its CR bundle agrees with the push-forward $\mathcal{Z}_*\mathcal{V}$. Conversely, if $\mathcal{M}$ is a CR submanifold of $\mathbb{C}^m$ and $\mathcal{V}$ is its CR bundle, then $(\mathcal{M},\mathcal{V})$ defines an integrable CR structure (see [BER] and [J] for more details).

In an abstract CR manifold $(\mathcal{M},\mathcal{V})$, a smooth section of $\mathcal{V}$ is called a *CR vector field*. A function f on $\mathcal{M}$ is called a *CR function* if $Lf = 0$ for any CR vector field L. The maximum principle for the modulus of CR functions when $(\mathcal{M},\mathcal{V})$ is embeddable has been studied by several authors (see e.g. [Ba; Ber; EHS; Io; Ro; Si]). To our knowledge, very little seems to be known when $(\mathcal{M},\mathcal{V})$ is not necessarily embeddable. The authors of [HNa] have proved a weak maximum principle for almost complex manifolds under some assumptions on the Levi form and minimality of the manifold (see [BER, p. 20]). When $(\mathcal{M},\mathcal{V})$ is locally embeddable, it

Received September 30, 2005. Revision received May 23, 2006.
Work supported in part by NSF Grant no. INT-0203005.