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On Manin's Conjecture for Singular del Pezzo Surfaces of Degree 4, I

R. de la Bretèche & T. D. Browning

1. Introduction

Let $Q_1, Q_2 \in \mathbb{Z}[x_0, \dots, x_4]$ be a pair of quadratic forms whose common zero locus defines a geometrically integral surface $X \subset \mathbb{P}^4$. Then X is a del Pezzo surface of degree 4. We assume henceforth that the set $X(\mathbb{Q}) = X \cap \mathbb{P}^4(\mathbb{Q})$ of rational points on X is nonempty, so that in particular $X(\mathbb{Q})$ is dense in X under the Zariski topology. Given a point $x = [x_0, \dots, x_4] \in \mathbb{P}^4(\mathbb{Q})$ with $x_0, \dots, x_4 \in \mathbb{Z}$ such that $\gcd(x_0, \dots, x_4) = 1$, we let $H(x) = \max_{0 \le i \le 4} |x_i|$. Then $H : \mathbb{P}^4(\mathbb{Q}) \to \mathbb{R}_{\ge 0}$ is the height attached to the anticanonical divisor $-K_X$ on X parametrized by the choice of norm $\max_{0 \le i \le 4} |x_i|$. A finer notion of density is provided by analyzing the asymptotic behavior of the quantity

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}) : H(x) \le B\},$$

as $B \to \infty$, for appropriate open subsets $U \subseteq X$. Since every quartic del Pezzo surface X contains a line, it is natural to estimate $N_{U,H}(B)$ for the open subset U obtained by deleting the lines from X. The motivation behind this paper is to consider the asymptotic behavior of $N_{U,H}(B)$ for singular del Pezzo surfaces of degree 4.

A classification of quartic del Pezzo surfaces $X \subset \mathbb{P}^4$ can be found in the work of Hodge and Pedoe [8, Book IV, Sec. XIII.11], which shows in particular that there are only finitely many isomorphism classes to consider. Let $\tilde{X}$ denote the minimal desingularization of X, and let $\operatorname{Pic} \tilde{X}$ be the Picard group of $\tilde{X}$. Then Manin has stated a very general conjecture [6] that predicts the asymptotic behavior of counting functions associated to suitable Fano varieties. In our setting this leads us to expect the existence of a positive constant $c_{X,H}$ such that

$$N_{U,H}(B) = c_{X,H} B (\log B)^{\rho - 1}(1 + o(1)), \tag{1.1}$$

as $B \to \infty$, where ρ denotes the rank of $\operatorname{Pic} \tilde{X}$. The constant $c_{X,H}$ has received a conjectural interpretation at the hands of Peyre [9], which in turn has been generalized to cover certain other cases by Batyrev and Tschinkel [2] and Salberger [11]. A brief discussion of $c_{X,H}$ will take place in Section 2.

There has been rather little progress towards the Manin conjecture for del Pezzo surfaces of degree 4. The main successes in this direction are to be found in work of Batyrev and Tschinkel [1], covering the case of toric varieties, and in the work of

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