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Limiting Weak-type Behavior for the Riesz Transform and Maximal Operator When $\lambda \to \infty$

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1. Introduction

The goal of this paper is to analyze the limiting weak-type behavior of important operators in harmonic analysis when they act on singular measures in $\mathbb{R}^n$. Consider the jth Riesz transform R_j defined on appropriate functions by

$$R_j f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \text{ p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy. \tag{1.1}$$

Here R_j is bounded from $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$ and from $L^1(\mathbb{R}^n)$ into the weak-L^1 space $L^{1,\infty}(\mathbb{R}^n)$. That is, there exist constants C_p for each $1 < p < \infty$ and C_1 such that, for all functions $f \in L^p(\mathbb{R}^n)$,

$$\|R_j f\|_p \le C_p \|f\|_p; \tag{1.2}$$

moreover, for all $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$,

$$\lambda m\{x \in \mathbb{R}^n : |R_j f(x)| > \lambda\} \le C_1 \|f\|_1. \tag{1.3}$$

These are referred to as the strong-type (p,p) and weak-type $(1,1)$ inequalities, respectively. See Stein [12] for the basic theory.

The strong-type (p,p) constant C_p is

$$C_p = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \le 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } 2 \le p < \infty. \end{cases}$$

This is proved by Pichorides [11] for $n = 1$ and completed by Iwaniec and Martin [5] for higher dimensions. When $n = 1$, the weak-type constant C_1 is

$$C_1 = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots}.$$

This is proved by Davis [2] and Baernstein [1]. However, for higher dimensions the question remains open. One conjecture regarding the weak-type constant is that it is independent of dimension n. A recent result [6] proved by the present

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