Michigan Math. J. 55 (2007)

A Finitely Presented Solvable Group with a Small Quasi-Isometry Group

Kevin Wortman

0. Introduction

Let $\mathbf{B}_n$ be the upper triangular subgroup of $\mathbf{SL}_n$, and let $\mathbf{Ad}(\mathbf{B}_n) \leq \mathbf{PGL}_n$ be the image of $\mathbf{B}_n$ under the map $\mathbf{Ad}\colon \mathbf{SL}_n \to \mathbf{PGL}_n$.

If p is a prime number, then the group $\mathbf{B}_n(\mathbb{Z}[1/p])$ is finitely presented for all $n \geq 2$. In particular it is finitely generated, so we can form its quasi-isometry group—denoted $\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p]))$. In this paper we shall prove the following result.

Theorem 1. *If* $n \geq 3$, *then*

$$\mathcal{QI}(\mathbf{B}_n(\mathbb{Z}[1/p])) \cong (\mathbf{Ad}(\mathbf{B}_n)(\mathbb{R}) \times \mathbf{Ad}(\mathbf{B}_n)(\mathbb{Q}_p)) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

The $\mathbb{Z}/2\mathbb{Z}$-action defining the semi-direct product in Theorem 1 is given by a $\mathbb{Q}$-automorphism of $\mathbf{PGL}_n$ that stabilizes $\mathbf{Ad}(\mathbf{B}_n)$. The order-2 automorphism acts simultaneously on each factor.

For the proof of Theorem 1, the reader may advance directly to Section 1. The proof continues in Sections 2 and 3.

0.1. What's New about This Example

The group $\mathbf{B}_n(\mathbb{Z}[1/p])$ is solvable, and when $n \geq 3$ its quasi-isometry group is "small" in two senses: (i) it is virtually a product of finite-dimensional Lie groups (real and p-adic), and (ii) it is solvable. The *quasi-isometry group* of a finitely generated group Γ is the group of all quasi-isometries of Γ modulo those that have finite distance in the sup-norm to the identity.

We don't know too much about which groups can be realized as quasi-isometry groups, but the examples we do have suggest this collection could be quite diverse. A fair amount of variety is displayed even in the extremely restricted class of finitely generated groups that appear as lattices in semisimple groups. These quasi-isometry groups include the $\mathbb{Q}$-, $\mathbb{R}$-, or $\mathbb{Q}_p$-points of simple algebraic groups; discrete groups that are finite extensions of lattices; and infinite-dimensional groups (for a complete list of references to these results see e.g. [Fa] or [Wo]).

However, the same variety is not presently visible in the class of quasi-isometry groups for infinite finitely generated amenable groups. In fact, the quasi-isometry

Received July 20, 2005. Revision received January 26, 2007.
Supported in part by an N.S.F. Postdoctoral Fellowship.