Michigan Math. J. 54 (2006)

Exceptional Values in Holomorphic Families of Entire Functions

Alexandre Eremenko

In 1926, Julia [5] studied singularities of implicit functions defined by equations $f(z, w) = 0$, where f is an entire function of two variables such that $f(\cdot, w) \not\equiv 0$ for every $w \in \mathbf{C}$. Among other things, he investigated the exceptional set P consisting of those w for which such an equation has no solutions z. In other words, P is the complement of the projection of the analytic set $\{(z, w) : f(z, w) = 0\}$ onto the second coordinate. Julia proved that P is closed and cannot contain a continuum unless it coincides with the w-plane. Lelong [6] and Tsuji [12; 13, Thm. VIII.37] independently improved this result by showing that the logarithmic capacity of P is zero if $P \neq \mathbf{C}$. In the opposite direction, Julia [5] proved that every discrete set $P \subset \mathbf{C}$ can occur as the exceptional set. He writes: "Resterait à voir si cet ensemble, sans être continu, peut avoir la puissance du continu." (It remains to be seen whether this set, without being a continuum, can have the power of a continuum.)

According to Alan Sokal (private communication), the same question arises in holomorphic dynamics when one tries to extend to holomorphic families of transcendental entire functions a result of Lyubich [8, Prop. 3.5] on holomorphic families of rational functions.

In this paper we show that, in general, the result of Lelong and Tsuji is best possible: every closed set of zero capacity can occur as an exceptional set (Theorem 1). Then we study a related problem of dependence of Picard exceptional values of the function $z \mapsto f(z, w)$ on the parameter w (Theorem 2).

It is known that the exceptional set P is discrete in the important case where $z \mapsto f(z, w)$ are functions of finite order. This was discovered by Lelong in [6]; the result was later generalized to the case of a multidimensional parameter w in [7, Thm. 3.44].

We also mention that the set P must be analytic in certain holomorphic families of entire functions with finitely many singular values, a situation that was considered in [1; 2]. These families may consist of functions of infinite order.

We begin with a simple proof of a version of Lelong's theorem on functions of finite order.

Proposition 1. *Let D be a complex manifold and $f : \mathbf{C} \times D \to \mathbf{C}$ an analytic function such that the entire functions $z \mapsto f(z, w)$ are not identically equal to zero and are of finite order for all $w \in D$. Then the set*

Received August 29, 2005. Revision received September 30, 2005.
The author was supported by NSF Grant nos. DMS-0100512 and DMS-0244547.