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Periodicities in Linear Fractional Recurrences: Degree Growth of Birational Surface Maps

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0. Introduction

Given complex numbers $\alpha_0, \dots, \alpha_p$ and $\beta_0, \dots, \beta_p$, we consider the recurrence relation

$$x_{n+p+1} = \frac{\alpha_0 + \alpha_1 x_{n+1} + \dots + \alpha_p x_{n+p}}{\beta_0 + \beta_1 x_{n+1} + \dots + \beta_p x_{n+p}}. \tag{0.1}$$

Thus a p-tuple $(x_1, \dots, x_p)$ generates an infinite sequence (x_n). We consider two equivalent reformulations in terms of rational mappings: we may consider the mapping $f : \mathbf{C}^p \to \mathbf{C}^p$ given by

$$f(x_1, \dots, x_p) = \left(x_2, \dots, x_p, \frac{\alpha_0 + \alpha_1 x_1 + \dots + \alpha_p x_p}{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}\right); \tag{0.2}$$

or we may use the imbedding $(x_1, \dots, x_p) \mapsto [1 : x_1 : \dots : x_p] \in \mathbf{P}^p$ into projective space and consider the induced map $f : \mathbf{P}^p \to \mathbf{P}^p$ given by

$$f_{\alpha,\beta}[x_0 : x_1 : \dots : x_p] = [x_0 \beta \cdot x : x_2 \beta \cdot x : \dots : x_p \beta \cdot x : x_0 \alpha \cdot x], \tag{0.3}$$

where $\alpha \cdot x = \alpha_0 x_0 + \dots + \alpha_p x_p$.

Here we will study the degree growth of the iterates $f^k = f \circ \dots \circ f$ of f. In particular, we are interested in the quantity

$$\delta(\alpha, \beta) := \lim_{k \to \infty} (\text{degree}(f^k_{\alpha,\beta}))^{1/k}.$$

A natural question is: For what values of α and β can (0.1) generate a periodic recurrence? In other words, when does (0.1) generate a periodic sequence (x_n) for all choices of $x_1, \dots, x_p$? This is equivalent to asking when there is an N such that $f^N_{\alpha,\beta}$ is the identity map. Periodicities in recurrences of the form (0.1) have been studied in [CLa; GrL; KoL; KGo; Ly]. The question of determining the parameter values α and β for which $f_{\alpha,\beta}$ is periodic has been known for some time and is posed explicitly in [GKP] and [GrL, p. 161]. Recent progress in this direction has been obtained in [CLa]. The connection with our work here is that, if $\delta(\alpha, \beta) > 1$, then the degrees of the iterates of $f_{\alpha,\beta}$ grow exponentially and $f_{\alpha,\beta}$ is far from periodic.

In the case $p = 1$, f is a linear (fractional) map of $\mathbf{P}^1$. The question of periodicity for f is equivalent to determining when a 2×2 matrix is a root of the identity.

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