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A Monotonicity Result for Volumes of Holomorphic Images

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Introduction

This paper studies the Euclidean $2k$-dimensional volumes of parameterized holomorphic images of certain pseudoconvex domains in complex Euclidean space $\mathbf{C}^k$. Its motivation arises from studying a more specific situation—namely, the classification problem for proper holomorphic mappings between balls in complex Euclidean spaces (usually of different dimensions). We provide two versions of a monotonicity result. The first applies for mappings from balls and eggs; its proof involves the computation of explicit integrals. The second applies for mappings from more general domains, but it requires some regularity assumptions on the mappings at the boundary.

By way of introduction, first suppose that $k = 1$, that f is holomorphic on the unit disk B_1, and that f' is square integrable. Then the area of the parameterized image of the unit disk is $\pi \|f'\|_{L^2}^2$. In particular, when $f(z) = z^m$, the area (with multiplicity taken into account) is $m\pi$. Furthermore, one can easily see that $\|(zf)'\|_{L^2}^2 \geq \|f'\|_{L^2}^2$ with strict inequality unless $f = 0$, so the area is increased when f is replaced by zf.

In this paper we generalize these results to the more complicated situation in higher dimensions and show how they fit into the classification problem. The main results are Theorem 3, its consequence Corollary 2, and Theorem 4. Theorem 3 states that parameterized volumes of images of balls and eggs increase under a tensor product operation; Corollary 2 provides a sharp upper bound for the parameterized volume of a proper polynomial mapping between balls in terms of its degree and the domain dimension. Theorem 4 provides a monotonicity result for more general domains, assuming that the mapping is continuously differentiable at the boundary so that Stokes's theorem can be applied.

Section I considers the 1-dimensional case. The tensor product $z \to z^{\otimes m}$ provides a natural generalization to higher dimensions of the map $z \to z^m$ in one dimension. We let H_m denote a concrete form of the mapping $z \to z^{\otimes m}$; see (6) in Section II. We show there how the mappings H_m combine with a *tensor product on a subspace* operation to play a crucial role in the classification of proper polynomial mappings between balls.

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