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Coniveau and the Grothendieck Group of Varieties

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There are two natural filtrations on the singular cohomology of a complex smooth projective variety: the coniveau filtration, which is defined geometrically; and the level filtration, which is defined Hodge theoretically. We will say that the generalized Hodge conjecture (GHC) holds for a variety X if these filtrations coincide on its cohomology. There are a number of intermediate forms of this condition, including the statement that the ordinary Hodge conjecture holds for X. We show that if the GHC (or an intermediate version of it) holds for X then it holds for any variety Y that defines the same class in a completion of a Grothendieck group of varieties. In particular, by using motivic integration we can see that this is the case if X and Y are birationally equivalent Calabi–Yau varieties or, more generally, K-equivalent varieties. This refines a result obtained in [A] by a different method.

The key point is to show that the singular cohomology with its coniveau (resp. level) filtration determines a homomorphism ν (resp. λ) from the Grothendieck group of varieties $\mathrm{K}_0(\mathcal{V}ar_{\mathbb{C}})$ to the Grothendieck group of polarizable filtered Hodge structures $\mathrm{K}_0(\mathcal{FHS})$. This is done by showing that cohomology together with these filtrations behaves appropriately under blowups. We then show that X satisfies GHC if and only if its class $[X]$ lies in the kernel of the difference $\nu - \lambda$, and the results just described follow from this.

The following conventions will be used throughout the paper. All our varieties will be defined over $\mathbb{C}$. We denote the singular cohomology of a smooth projective variety X with rational coefficients by $H^i(X)$. Our thanks to the referee for a number of helpful suggestions.

1. Filtered Hodge Structures

Let X be a smooth projective variety. Its cohomology carries a natural Hodge structure. The *coniveau* filtration on $H^i(X)$ is given by

$$N^pH^i(X) = \sum_{\operatorname{codim} S \ge p} \ker[H^i(X) \to H^i(X - S)],$$

which is a descending filtration by sub-Hodge structures. The largest rational sub-Hodge structure $\mathcal{F}^pH^i(X)$ contained in $F^pH^i(X)$ gives a second filtration, which we call the *level filtration*. We have $N^pH^i(X) \subseteq \mathcal{F}^pH^i(X)$. We will say that

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