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Degenerations and Fundamental Groups Related to Some Special Toric Varieties

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1. Introduction

Let X be a projective algebraic surface embedded in a projective space $\mathbb{CP}^N$. Take a general linear subspace V in $\mathbb{CP}^N$ of dimension $N-3$. Then the projection centered at V to $\mathbb{CP}^2$ defines a finite map $f: X \to \mathbb{CP}^2$. Let $B \subset \mathbb{CP}^2$ be the branch curve of f, and let $\pi_1(\mathbb{CP}^2 \setminus B)$ be *the fundamental group of the complement of the branch curve*. This group is an invariant of the surface. Closely related to this group is the affine part $\pi_1(\mathbb{C}^2 \setminus B)$.

In this work we compute the groups just defined as they relate to four toric varieties. The first surface is $X_1 := F_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$, the Hirzebruch surface of degree 1 in $\mathbb{CP}^6$ embedded by the line bundle with the class $s + 3g$, where s is the negative section and g is a general fiber. The second surface is $X_2 := F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$, the Hirzebruch surface of degree 0 in $\mathbb{CP}^7$ embedded by $\mathcal{O}(1,3)$; we generalize the results to the case where X_2 is embedded in $\mathbb{CP}^{2n+1}$ by $\mathcal{O}(1,n)$. The third is $X_3 := F_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ in $\mathbb{CP}^5$ embedded by the class $s + 3g$, and the fourth is a singular toric surface X_4 with one A_1 singular point embedded in $\mathbb{CP}^6$. Here A_1-singularity is an isolated normal singularity of dimension 2 whose resolution consists of one (-2)-curve (i.e., a nonsingular rational curve on a surface with -2 as its self-intersection number). For the first three cases, we use different triangulations of tetragons from those treated in [24] and [25].

This work fits into the program initiated by Moishezon and Teicher to study complex surfaces via braid monodromy techniques. They defined the generators of a braid group from a line arrangement in $\mathbb{CP}^2$, which is the branch curve of a generic projection from a union of projective planes [24]—namely, degeneration. In order to explain the process of such a degeneration, they used schematic figures consisting of triangulations of triangles and tetragons [20; 23; 24]. Moishezon and Teicher studied the cases where X is the projective plane embedded by $\mathcal{O}(3)$

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