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Simply Connected Constant Mean Curvature Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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1. Introduction

In [9] Meeks proved that, if M is a properly embedded simply connected surface of constant mean curvature $H \neq 0$ in $\mathbb{R}^3$, then M is a round sphere. In particular, M cannot be topologically $\mathbb{R}^2$. More generally, he proved there is no properly embedded H-surface of finite topology in $\mathbb{R}^3$ with exactly one end. Afterwards, in [7] a different proof of Meeks's theorem was found, and in [6] it was extended to the hyperbolic space $\mathbb{H}^3$.

In this paper we consider this problem in $\mathbb{H}^2 \times \mathbb{R}$. There are properly embedded H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that are topologically $\mathbb{R}^2$; there are entire graphs (vertical graphs over $\mathbb{H}^2$) for each H, $0 \leq H \leq 1/2$ (see [10; 11]). We will prove that such a surface cannot exist for $H > 1/\sqrt{3}$. More generally, we prove the following statement.

Theorem 1.1. *For $H > 1/\sqrt{3}$, there is no properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.*

Hsiang and Hsiang showed that any compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ is a rotational sphere and has mean curvature greater than $1/2$ (see [5; 11]). Abresch and Rosenberg proved that, if the surface is simply connected, then the same result holds for compact H-surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$ (see [1]).

It is interesting to consider to what extent Theorem 1.1 holds in other homogeneous 3-manifolds (for some constant other than $1/\sqrt{3}$). In $\mathbb{S}^2 \times \mathbb{R}$ there is no properly embedded H-surface with one end. To see this, observe that an end of such a surface M would have to go up or down (but not both), since M is proper. Hence one can assume M is bounded below by height 0, say. Then Alexandrov reflection with respect to the "planes" $\mathbb{S}^2 \times \{t\}$ coming up from $t = 0$ allows us to conclude that the part of M below any $M \times \{t\}$ is a vertical graph. This contradicts the height estimates for such graphs (see [4]), so no such M exists in $\mathbb{S}^2 \times \mathbb{R}$.

The other homogeneous 3-manifolds (beside the space forms) are the Berger spheres, Heisenberg space, and $\widetilde{PSL}(2, \mathbb{R})$. Since the Berger spheres are compact, the question is interesting only for the Heisenberg space and $\widetilde{PSL}(2, \mathbb{R})$. Another

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