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Derived Categories of Toric Varieties

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1. Introduction

The purpose of this paper is to investigate the structure of the derived category of a toric variety. We shall prove the following result.

Theorem 1.1. *Let X be a projective toric variety with at most quotient singularities, let B be an invariant $\mathbb{Q}$-divisor whose coefficients belong to the set $\left\{\frac{r-1}{r}; r \in \mathbb{Z}_{>0}\right\}$, and let $\mathcal{X}$ be the smooth Deligne–Mumford stack associated to the pair (X, B) as in* [12]. *Then the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(\mathcal{X}))$ has a complete exceptional collection consisting of sheaves.*

An object of a triangulated category $a \in T$ is called *exceptional* if

$$\mathrm{Hom}^p(e, e) \cong \begin{cases} \mathbb{C} & \text{for } p = 0, \\ 0 & \text{for } p \neq 0. \end{cases}$$

A sequence of exceptional objects $\{e_1, \dots, e_m\}$ is said to be an *exceptional collection* if

$$\mathrm{Hom}^p(e_i, e_j) = 0 \quad \text{for all } p \text{ and } i > j.$$

The sequence is said to be *strong* if, in addition, $\mathrm{Hom}^p(e_i, e_j) = 0$ for $p \neq 0$ and all i, j; it is called *complete* if T coincides with the smallest triangulated subcategory containing all the e_i (cf. [2]).

It is usually hard to determine the explicit structure of a derived category of a variety. But it is known that some special varieties, such as projective spaces or Grassmann varieties, have strong complete exceptional collections consisting of vector bundles [1; 8; 9; 10]. Such sheaves are useful for further investigation of the derived categories (see e.g. [6; 7; 14; 18]).

We use the minimal model program for toric varieties as developed in [17] (and corrected in [15]) in order to prove the theorem. A special feature of this approach is that, even if we start with the smooth and nonboundary case $B = 0$, we are forced to deal not only with singularities but also with the case $B \neq 0$ because Mori fiber spaces have multiple fibers in general. Thus we are inevitably led to consider the general situation concerning Deligne–Mumford stacks even if we need results for smooth varieties only. The "stacky" sheaves need careful treatment because there exist nontrivial stabilizer groups on the stacks (cf. Remark 5.1).

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