Michigan Math. J. 54 (2006)

GIT Equivalence beyond the Ample Cone

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Introduction

The approach to moduli spaces (e.g., for curves of fixed genus) presented by D. Mumford in his geometric invariant theory [17] relies on his construction of quotients for actions of reductive groups G on algebraic varieties X. He introduces the notion of a G-linearized line bundle on X, and to any such bundle L he associates a G-invariant open set $X^{ss}(L) \subset X$ of semistable points. This set admits a so-called good quotient $X^{ss}(L) \to X^{ss}(L)/\!\!/G$ with a quasiprojective quotient space.

Mumford's construction, however, is in general not unique: his "GIT quotients" turn out to depend essentially on the choice of the bundle and the linearization. Therefore, it is a natural desire to describe the collection of all possible GIT quotients for a given reductive group action. For "ample GIT quotients"—that is, those arising from linearized ample line bundles—this problem has been studied by several authors; see [8; 10; 22] and [19].

A first basic step in the study of ample GIT quotients is to show that there are only finitely many of them (see [5; 10; 20; 22]). Then the subject becomes combinatorial. The situation is described by a sort of fan subdividing the so-called (open) G-ample cone: the cones of this fan correspond to the ample GIT quotients and the face relations reflect, in an order-reversing manner, the set-theoretical inclusion of the respective sets of semistable points; see [19].

However, there are interesting examples of projective GIT quotients that do not arise from linearized ample bundles (see [6]). Motivated by this observation, we study here the situation beyond the G-ample cone, and we propose a combinatorial framework for the description of the phenomena occurring there. We restrict our attention to the case of a torus action. This case is the most vivid one concerning variation of GIT quotients, and it allows an elementary treatment.

The setup is as follows: X is a normal projective variety over an algebraically closed field $\mathbb{K}$ of characteristic zero such that X has a free finitely generated divisor class group $\mathrm{Cl}(X)$ as well as a finitely generated total coordinate ring (see Section 3)

$$\mathcal{R}(X) := \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)).$$

Received March 7, 2005. Revision received April 24, 2006.