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A Conformal Invariant for Nonvanishing Analytic Functions and Its Applications

Yong Chan Kim & Toshiyuki Sugawa

Dedicated to Professor Yukio Kusunoki on the occasion of his 80th birthday

1. Introduction

Conformal invariants play a central role in the modern theory of functions of a complex variable. One of the most important invariants is the hyperbolic metric $\rho_D(z)|dz|$ of a hyperbolic plane domain D. Recall that a subdomain D of $\mathbb{C}$ is called *hyperbolic* if D admits an analytic universal covering projection p of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto D. Then the density $\rho_D(z)$ of a hyperbolic metric is defined by the equation $\rho_D(z)|p'(\zeta)| = 1/(1-|\zeta|^2)$ for $\zeta \in p^{-1}(z)$. Note that the value of $\rho_D(z)$ does not depend on the particular choice of ζ or p. The Poincaré–Koebe uniformization theorem tells us that $D \subset \mathbb{C}$ is hyperbolic if and only if D is neither the whole plane $\mathbb{C}$ nor the punctured plane $\mathbb{C} \setminus \{a\}$ for any $a \in \mathbb{C}$. The hyperbolic metric is conformally invariant in the sense that the pulled-back density $f^*\rho_{D'}(z) = \rho_{D'}(f(z))|f'(z)|$ of $\rho_{D'}(w)|dw|$ under a conformal map $f\colon D \to D'$ is equal to $\rho_D(z)$. Throughout the paper, a conformal map means a conformal homeomorphism.

In this paper we propose a sort of conformal invariant associated with a nonvanishing analytic function. This quantity proves its usefulness in estimating the hyperbolic sup-norm of the pre-Schwarzian derivative of a locally univalent function in various situations (cf. [15]). Let φ be a nonvanishing analytic function on a hyperbolic domain D; namely, $\varphi\colon D \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is holomorphic. Then we set

$$V_D(\varphi) = \sup_{z \in D} \rho_D(z)^{-1} \left| \frac{\varphi'(z)}{\varphi(z)} \right|.$$

This quantity measures the rate of growth of φ compared with the hyperbolic metric. Note also that $V_D(\varphi)$ can be thought of as the Bloch seminorm of the (possibly multivalued) function $\log \varphi$. The quantity $V_D(\varphi)$ does not depend on the source domain D; more precisely, $V_{D_0}(\varphi \circ f) = V_D(\varphi)$ for a conformal map $f\colon D_0 \to D$ (see Theorem 2.2). On the other hand, $V_D(\varphi)$ may depend on the target domain.

One merit of this quantity is monotonicity in several respects. For instance, if ω is a holomorphic map of D_0 into D then $V_D(\varphi) \le V_{D_0}(\varphi \circ \omega)$ holds (see Theorem 2.2). Many more properties will be discussed in Section 2.

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