Michigan Math. J. 54 (2006)

Coincident Root Loci of Binary Forms

L. M. Fehér, A. Némethi, & R. Rimányi

1. Introduction

Coincident root loci are subvarieties of $S^d\mathbb{C}^2$—the space of binary forms of degree d—labeled by partitions of d. Given a partition λ, let X_λ be the set of forms with root multiplicity corresponding to λ. There is a natural action of $\mathrm{GL}_2(\mathbb{C})$ on $S^d\mathbb{C}^2$, and the coincident root loci are invariant under this action. We calculate their equivariant Poincaré duals, generalizing formulas of Hilbert and Kirwan. In the second part we apply these results to present the cohomology ring of the corresponding moduli spaces (in the GIT sense) by geometrically defined relations.

One of the main goals of geometric invariant theory is to calculate the cohomology ring of a geometric quotient. For the case when all semistable point are stable, several techniques have been developed. But even for very simple representations this condition is not satisfied. In this paper we study the action of $\mathrm{GL}(2)$ on the space of binary forms of degree d. In the case of d odd the methods of [JK; K1; M] can be applied, but none of these methods computes the cohomology ring of the moduli space in the case of d even. We show how equivariant Poincaré-dual calculations lead to relations for the cohomology ring in both the odd and the even case.

Closely related rings have been computed previously. The computation for $H^*_G(X^{ss})$ is well known (since Kirwan's thesis in the case of Betti numbers), and the existing procedure is independent of d being even or odd. In the d even case, rational intersection cohomology of the moduli space is also known [K2], a result we also recover in Remark 4.11.

Our Poincaré-dual (a.k.a. Thom polynomial) calculations are also interesting in their own right because they generalize formulas of Hilbert and Kirwan on coincident root loci. These calculations not only lead to explicit relations for these cohomology rings but also identify them with the equivariant Poincaré-duals of the simplest unstable coincident root loci.

Consider the dth symmetric power $S^d\mathbb{C}^2$ of the standard representation of $\mathrm{GL}_2(\mathbb{C})$—that is, the action of G on the space V_d of degree-d homogeneous polynomials in two variables x, y. For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of d (i.e., $\sum_j \lambda_j = d$) we define

Received February 24, 2005. Revision received February 16, 2006.

Supported by OTKA grants (1st and 3rd author T046365MAT); NSF grant (2nd author DMS-0088950, 3rd author DMS-0405723); and Marie Curie grant (2nd author).