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Tight Closure Test Exponents for Certain Parameter Ideals

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0. Introduction

Throughout the paper, R will denote a commutative Noetherian ring of prime characteristic p. We shall always denote by $f: R \to R$ the Frobenius homomorphism, for which $f(r) = r^p$ for all $r \in R$. Let $\mathfrak{a}$ be an ideal of R. The nth *Frobenius power* $\mathfrak{a}^{[p^n]}$ of $\mathfrak{a}$ is the ideal of R generated by all p^nth powers of elements of $\mathfrak{a}$.

We use R° to denote the complement in R of the union of the minimal prime ideals of R. An element $r \in R$ belongs to the *tight closure* $\mathfrak{a}^*$ *of* $\mathfrak{a}$ if and only if there exists a $c \in R^\circ$ such that $cr^{p^n} \in \mathfrak{a}^{[p^n]}$ for all $n \gg 0$. We say that $\mathfrak{a}$ is *tightly closed* precisely when $\mathfrak{a}^* = \mathfrak{a}$. The theory of tight closure was invented by M. Hochster and C. Huneke [4], and many applications have been found for the theory (see [7]). For the definition of the tight closure N_M^* of a submodule N in an ambient R-module M (and explanation of the notations $N_M^{[p^n]}$ and m^{p^n} for $m \in M$ and a nonnegative integer n), the reader is referred to [4, (8.1)–(8.3)].

A p^{w_0}*-weak test element* for R (where w_0 is a nonnegative integer) is an element $c' \in R^\circ$ such that, for every finitely generated R-module M and every submodule N of M and for $m \in M$, we have $m \in N_M^*$ if and only if $c'm^{p^n} \in N_M^{[p^n]}$ for all $n \geq w_0$. A p^0-weak test element is called a *test element*. A *locally stable* p^{w_0}*-weak test element* (respectively, *completely stable* p^{w_0}*-weak test element*) for R is an element $c' \in R$ such that, for every prime ideal $\mathfrak{p}$ of R, the natural image $c'/1$ of c' in the localization $R_\mathfrak{p}$ is a p^{w_0}-weak test element for $R_\mathfrak{p}$ (respectively, for the completion $\widehat{R_\mathfrak{p}}$ of $R_\mathfrak{p}$). When $w_0 = 0$, we omit the adjective "p^{w_0}-weak". A locally stable p^{w_0}-weak test element for R is a p^{w_0}-weak test element for R, and a completely stable p^{w_0}-weak test element for R is a locally stable p^{w_0}-weak test element for R; see [4, Prop. (8.13)].

It is a result of Hochster and Huneke [5, Thm. (6.1)(b)] that an algebra of finite type over an excellent local ring of characteristic p has a completely stable p^{w_0}-weak test element for some w_0; furthermore, such an algebra that is also reduced actually has a completely stable test element.

This paper is concerned with the concept of a *test exponent* in tight closure theory introduced by Hochster and Huneke in [6, Def. 2.2]. Let c be a test element

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