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On the Local Behavior of the Carmichael λ-Function

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1. Introduction

Let ϕ denote the *Euler function,* which, for an integer $n \geq 1$, is defined as usual by

$$\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^{\times} = \prod_{p^{\nu} \| n} p^{\nu-1}(p-1).$$

The *Carmichael function* λ is defined for each integer $n \geq 1$ as the largest order of any element in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^{\times}$. More explicitly, for any prime power p^{ν} we have:

$$\lambda(p^{\nu}) = \begin{cases} p^{\nu-1}(p-1) & \text{if } p \geq 3 \text{ or } \nu \leq 2, \\ 2^{\nu-2} & \text{if } p = 2 \text{ and } \nu \geq 3; \end{cases}$$

and, for an arbitrary integer $n \geq 2$,

$$\lambda(n) = \operatorname{lcm}[\lambda(p_1^{\nu_1}), \dots, \lambda(p_k^{\nu_k})],$$

where $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$ is the prime factorization of n. Note that $\lambda(1) = 1$.

For a positive integer n, let $\Omega(n)$, $\omega(n)$, $\tau(n)$, and $\sigma(n)$ denote (respectively) the number of prime divisors of n with and without repetitions, the total number of divisors of n, and their sum. Let f be any one of the functions Ω, ω, τ, ϕ, or σ. It is well known that, if t is any positive integer and a is any permutation of $\{1, \dots, t\}$, then there exist infinitely many positive integers n such that all inequalities $f(n + a(i)) > f(n + a(i + 1))$ hold for $i = 1, \dots, t - 1$. In fact, in [3] it is shown that, if a, b are any two permutations of $\{1, \dots, t\}$, then there exist infinitely many positive integers n such that all inequalities $\omega(n + a(i)) > \omega(n + a(i + 1))$ and $\tau(n + b(i)) > \tau(n + b(i + 1))$ hold for $i = 1, \dots, t - 1$.

In this note, we prove some effective versions of this result from [3] with the pair of functions $\{\omega, \tau\}$ replaced by the pair $\{\lambda, \phi\}$.

We use the Vinogradov symbols $\gg$, $\ll$, and $\asymp$ as well as the Landau symbols O and o with their usual meaning. We use the letters p and q for prime numbers. For a positive real number x we write $\log_1 x = \max\{1, \log x\}$, where log is the natural logarithm, and for a positive integer $k \geq 2$ we define $\log_k x = \log_1(\log_{k-1} x)$. When $k = 1$, we omit the subscript and thus understand that all the logarithms that will appear are ≥ 1. We write $\pi(x)$ for the number of primes $p \leq x$ and

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