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On Some Lacunary Power Series

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1. Introduction

Consider a lacunary power series given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^{k_n}, \tag{1}$$

where $k_{n+1}/k_n \geq b > 1$ for every $n \geq 0$ and where $a_n \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} |a_n| < \infty$. Then f is holomorphic in the unit disc $\mathbb{D}$ and continuous in $\bar{\mathbb{D}}$.

In 1945, Salem and Zygmund showed in [SZ] that if $b > b_0$ for a constant $b_0 \approx 45$ and if the a_n satisfy some conditions (so that the convergence of $\sum_{n=0}^{\infty} |a_n|$ is slow enough), then the image of the unit circle under f is a Peano curve—that is, it contains an open set in the plane. In 1963, Kahane, M. Weiss, and G. Weiss in [KWW] extended the result, showing that for every $b > 1$ there exists a constant $\gamma > 0$ depending only on b and such that, if

$$|a_n| \leq \gamma \sum_{m=n+1}^{\infty} |a_m| \tag{2}$$

for every n, then the image of the unit circle under f is a Peano curve. In fact, they proved that there exist constants K, ξ, ν (depending only on b) such that, if

- inequality (2) is fulfilled and
- E is any Cantor set in the unit circle obtained by taking an arc I of length at least ξ/k_0, removing the middle subarc of I of length K times the length of I and repeating the procedure inductively, always removing the middle subarc of length K times the length of the larger one,

then $f(E)$ contains the disc centered at 0 of radius $\nu \sum_{n=0}^{\infty} |a_n|$.

In [CGP] it was noticed by Cantón, Granados, and Pommerenke that the Kahane–Weiss–Weiss result implies the following.

CGP Theorem. *If f is a map of the form* (1) *satisfying* (2) *and if k_0 is sufficiently large, then f does not preserve Borel sets on the unit circle.*

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