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# Graded Cofinite Rings of Differential Operators

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## 1. Introduction

In this paper we study subalgebras $\mathcal{A}$ of the algebra $\mathcal{D}(X)$ of differential operators on a smooth variety $X$ that are big in the following sense: using the order of a differential operator, the ring $\mathcal{D}(X)$ is equipped with a filtration. Its associated graded algebra $\bar{\mathcal{D}}(X)$ is commutative and can be regarded as the set of regular functions on the cotangent bundle of $X$. The subalgebra $\mathcal{A}$ inherits a filtration from $\mathcal{D}(X)$, and its associated graded algebra $\bar{\mathcal{A}}$ is a subalgebra of $\bar{\mathcal{D}}(X)$. We call $\mathcal{A}$ *graded cofinite* in $\mathcal{D}(X)$ if $\bar{\mathcal{D}}(X)$ is a finitely generated $\bar{\mathcal{A}}$-module.

Our guiding example of a graded cofinite subalgebra is the algebra of invariants $\mathcal{D}(X)^W$, where $W$ is a finite group acting on $X$. Other examples can be constructed as follows. Let $\varphi\colon X \to Y$ be a finite dominant morphism onto a normal variety $Y$. Then we put

$$\mathcal{D}(X,Y) = \{D \in \mathcal{D}(X) \mid D(\mathcal{O}(Y)) \subseteq \mathcal{O}(Y)\}. \tag{1.1}$$

We show (Corollary 3.6) that this subalgebra is graded cofinite if and only if the ramification of $\varphi$ is uniform—that is, if the ramification degree of $\varphi$ along a divisor $D \subset X$ depends only on the image $\varphi(D)$.

It should be noted that these two constructions are in fact more or less equivalent. In Theorem 3.1 we show that $\mathcal{D}(X)^W = \mathcal{D}(X, X/W)$. Conversely, we show in Proposition 3.3 that $\mathcal{D}(X,Y) = \mathcal{D}(\tilde{X})^W$, where $\tilde{X} \to X$ is a suitable finite cover of $X$ and $W$ is a finite group acting on $\tilde{X}$.

Our main result is that, up to automorphisms, every graded cofinite subalgebra is of the form just described.

1.1. Theorem. *Let $X$ be a smooth variety and $\mathcal{A}$ a graded cofinite subalgebra of $\mathcal{D}(X)$. Then there is an automorphism $\Phi$ of $\mathcal{D}(X)$, inducing the identity on $\bar{\mathcal{D}}(X)$, such that $\mathcal{A} = \Phi\mathcal{D}(X,Y)$ for some uniformly ramified morphism $\varphi\colon X \to Y$.*

The main motivation for this notion came from the following result of Levasseur and Stafford [LSt]. Let $W$ be a finite group acting linearly on a vector space $V$; then $\mathcal{D}(V)^W$ is generated by the $W$-invariant functions $\mathcal{O}(V)^W$ and the $W$-invariant constant coefficient differential operators $S^*(V)^W$. For general varieties $X$, there

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