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Distributional Properties of the Largest Prime Factor

William D. Banks, Glyn Harman, & Igor E. Shparlinski

1. Introduction

For every positive integer n, let $P(n)$ denote the largest prime factor of n, with the usual convention that $P(1) = 1$. For an integer $q \ge 1$ and a real number z, we define $\mathbf{e}_q(z) = \mathbf{e}(z/q)$, where $\mathbf{e}(z) = \exp(2\pi i z)$ as usual.

In Section 3, we consider the problem of bounding the function

$$\varrho(x; q, a) = \#\{n \le x : P(n) \equiv a \pmod q\}.$$

For the case of q fixed, this question has been previously considered by Ivić [11]. However, the approach in [11] apparently does not extend to the case where the modulus q is allowed to grow with the parameter x; this is mainly due to the fact that asymptotic formulas for the number of primes in arithmetic progressions are much less precise for growing moduli than those known for a fixed modulus.

We also remark that Oon [13] has studied the distribution of $P(n)$ over the congruence classes of a fixed modulus q in the case of n itself belonging to an arithmetic progression (with a growing modulus).

In this paper, we use a similar approach to that of Ivić [11] and obtain new bounds that are nontrivial for a wide range of values of the parameter q. In particular, if q is not too large relative to x, we derive the expected asymptotic formula

$$\varrho(x; q, a) \sim \frac{x}{\varphi(q)}$$

with an explicit error term that is independent of a. On the other hand, we show that this estimate is no longer correct (even by an order of magnitude) for $q \ge \exp\bigl(3\sqrt{\log x \log\log x}\,\bigr)$.

In Section 4 we study the function

$$\varpi(x; q, a) = \#\{p \le x : P(p-1) \equiv a \pmod q\},$$

where p varies over the set of prime numbers, and we derive the upper bound

$$\varpi(x; q, a) \ll \frac{\pi(x)}{\varphi(q)}$$

provided that $\log q \le \log^{1/3} x$. Here, $\pi(x) = \#\{p \le x\}$. We expect that the matching lower bound $\varpi(x; q, a) \gg \pi(x)/\varphi(q)$ also holds for such q, or perhaps even

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