Michigan Math. J. 53 (2005)

Normality and Shared Functions of Holomorphic Functions and Their Derivatives

Jianming Chang & Mingliang Fang

1. Introduction

Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions defined in D. The family $\mathcal{F}$ is said to be *normal* in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ that converges, spherically locally uniformly in D, to a meromorphic function or to ∞ (see [7; 12; 14]).

Let f and g be meromorphic functions in a domain D in $\mathbb{C}$, and let a and b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = b$. If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = b$. If $f(z) = a \Leftrightarrow g(z) = a$ then we say that f and g *share* a in D.

Schwick [13] was the first to draw a connection between values shared by functions in $\mathcal{F}$ (and their derivatives) and the normality of the family $\mathcal{F}$. Specifically, he showed that if there exist three distinct complex numbers a_1, a_2, a_3 such that f and f' share a_j $(j = 1, 2, 3)$ in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D. Pang and Zalcman [10] extended this result as follows.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a, b, c, d be complex numbers such that $c \neq a$ and $d \neq b$. If for each $f \in \mathcal{F}$ we have $f(z) = a \Leftrightarrow f'(z) = b$ and $f(z) = c \Leftrightarrow f'(z) = d$, then $\mathcal{F}$ is normal in D.*

Chen and Hua proved the following.

Theorem B ([4], cf. [5; 9]). *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a $(\neq 0)$ be a finite complex value. If, f, f', and f'' share a in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In this paper, we extend Theorem B as follows.

Theorem 1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let $a(z)$ be an analytic function in D such that $a' \not\equiv a$. If, for each $f \in \mathcal{F}$, $f(z) = a(z) \Leftrightarrow f'(z) = a(z) \Leftrightarrow f''(z) = a(z)$ and $f(z) - a(z) = 0 \to f'(z) - a(z) = 0$ in D, then $\mathcal{F}$ is normal in D.*

Received July 22, 2004. Revision received January 28, 2005.

Supported by the NNSF of China (Grant no. 10471065), the NSFU of Jiangsu Province (Grant no. 04KJD110001), the SRF for ROCS, SEM., and the Presidential Foundation of South China Agricultural University.