Michigan Math. J. 53 (2005)

Magnus Intersections in One-Relator Products

James Howie

1. Introduction

Recall that a group G is *locally indicable* if each of its nontrivial, finitely generated subgroups admits an infinite cyclic homomorphic image. A *one-relator product* of groups A_λ ($\lambda \in \Lambda$) is a quotient

$$G = (*_{\lambda\in\Lambda} A_\lambda)/\langle\langle R\rangle\rangle$$

of their free product by the normal closure of a word R, which is called the *relator,* and is assumed not to be conjugate to an element of one of the A_λ. Let $A_\Lambda := *_{\lambda\in\Lambda} A_\lambda$. Then, by the Freiheitssatz for locally indicable groups [3] (see Theorem 2.1 to follow), a free factor $A_M := *_{\mu\in M} A_\mu$ of A_Λ embeds in G provided that R is not conjugate in A_Λ to an element of A_M. The image of this embedding is called the *Magnus subgroup* corresponding to the subset $M \subset \Lambda$.

The purpose of this paper is to examine the intersection of two Magnus subgroups of a one-relator product of locally indicable groups. Suppose M and N are two subsets of Λ. Then clearly the Magnus subgroup $A_{M\cap N}$ is contained in the intersection of Magnus subgroups $A_M \cap A_N$. In almost all cases it turns out that

$$A_M \cap A_N = A_{M\cap N},$$

but it is easy to construct examples where this equation fails.

In the special case where the A_λ are all infinite cyclic (so that G is a one-relator group), Collins [5] has proved the following result.

Theorem A. *Let A_M and A_N be Magnus subgroups of a one-relator group $G = A_\Lambda/\langle\langle R\rangle\rangle$. Then*

$$A_M \cap A_N = A_{M\cap N} * I,$$

where I is a free group of rank 0 *or* 1.

Both possibilities occur, but the most usual situation is that I has rank 0. If I has rank 1 then we say that A_M and A_N have *exceptional intersection.* In this case, nontrivial elements of I are called *exceptional elements.*

The purpose of this paper is twofold: to generalize Theorem A to the case of arbitrary locally indicable factors; and to investigate precisely under what circumstances an exceptional intersection can occur.

Received July 9, 2004. Revision received February 2, 2005.