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A Linear Bound for Frobenius Powers and an Inclusion Bound for Tight Closure

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Introduction

Let R denote a Noetherian ring, let $\mathfrak{m}$ denote a maximal ideal in R, and let I denote an $\mathfrak{m}$-primary ideal. This means by definition that $\mathfrak{m}$ is the radical of I. Then there exists a (minimal) number k such that $\mathfrak{m}^k \subseteq I \subseteq \mathfrak{m}$ holds. If R contains a field of positive characteristic p then the Frobenius powers of the ideal I, that is,

$$I^{[q]} = \{f^q : f \in I\}, \quad q = p^e,$$

are also $\mathfrak{m}$-primary and hence there exists a minimal number $k(q)$ such that $\mathfrak{m}^{k(q)} \subseteq I^{[q]}$ holds. In this paper we deal with the question of how $k(q)$ behaves as a function of q; in particular, we look for linear bounds for $k(q)$ from above. If $\mathfrak{m}^k \subseteq I$ and if l denotes the number of generators for $\mathfrak{m}^k$, then we obtain the trivial linear inclusion $(\mathfrak{m}^k)^{lq} \subseteq (\mathfrak{m}^k)^{[q]} \subseteq I^{[q]}$.

The main motivation for this question comes from the theory of tight closure. Recall that the tight closure of an ideal I in a domain R containing a field of positive characteristic p is the ideal

$$I^* = \{f \in R : \exists 0 \neq c \in R \text{ such that } cf^q \in I^{[q]} \text{ for all } q = p^e\}.$$

A linear inclusion relation $\mathfrak{m}^{\lambda q+\gamma} \subseteq I^{[q]}$ for all $q = p^e$ implies the inclusion $\mathfrak{m}^{\lambda} \subseteq I^*$, since then we can take any element $0 \neq c \in \mathfrak{m}^{\gamma}$ to show for $f \in \mathfrak{m}^{\lambda}$ that $cf^q \in \mathfrak{m}^{\lambda q+\gamma} \subseteq I^{[q]}$ and hence $f \in I^*$. The trivial bound mentioned previously yields $\mathfrak{m}^{kl} \subseteq I^*$, but this does not yield anything of interest because, in fact, we have already $\mathfrak{m}^{kl} \subseteq \mathfrak{m}^k \subseteq I$.

We restrict our attention in this paper to the case of a normal standard-graded domain R over an algebraically closed field $K = R_0$ of positive characteristic p and to a homogeneous R_+-primary ideal I. The question is then to find the minimal degree $k(q)$ such that $R_{\geq k(q)} \subseteq I^{[q]}$ or at least to find a good linear bound $k(q) \leq \lambda q + \gamma$. In this setting we work mainly over the normal projective variety $Y = \operatorname{Proj} R$ endowed with the very ample invertible sheaf $\mathcal{O}_Y(1)$. If $I = (f_1, \dots, f_n)$ is given by homogeneous ideal generators f_i of degree $d_i = \deg(f_i)$, then on Y we have the following short exact sequences of locally free sheaves:

$$0 \longrightarrow \operatorname{Syz}(f_1^q, \dots, f_n^q)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_Y(m - qd_i) \xrightarrow{f_1^q, \dots, f_n^q} \mathcal{O}_Y(m) \longrightarrow 0.$$

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