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Self-Maps of Projective Bundles on Projective Spaces

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Introduction

As shown in [3; 7; 9], the basic dynamical properties of the self-maps of projective spaces can be summarized as follows. Any holomorphic self-map $\mathbb{P}^r \xrightarrow{f} \mathbb{P}^r$ lifts through the canonical map $\mathbb{C}^{r+1} \setminus 0 \xrightarrow{q} \mathbb{P}^r$ to a self-map $\mathbb{C}^{r+1} \xrightarrow{F} \mathbb{C}^{r+1}$, $qF = fq$, whose components are homogeneous polynomials of degree d, the algebraic degree of f. When $d > 1$, the origin is a super-attracting fixed point for F with bounded and complete circular basin of attraction. The Green function associated to this basin is given by the formula $G = \lim_j \log\|F^j\|/d^j$, so it is plurisubharmonic. If $\mathcal{H}$ denotes the open set where G is pluriharmonic, then the Fatou set $\mathcal{F}$ of f equals $q(\mathcal{H})$. It follows that $\mathcal{F}$ is Stein and Kobayashi hyperbolic and that, when $r \geq 2$, the complement $\mathcal{J}$ of $\mathcal{F}$ is connected.

In this paper we extend these results to the context of projective bundles on projective manifolds. In the first part, we discuss the structure of the self-maps of a projective bundle $\mathbb{P}E \xrightarrow{p} B$ (fiber-degree, algebraic degree, completely invariant sub-bundles, dimension of the space of self-maps, lifting to E'). In the second part, we introduce Green functions and use them in the analysis of the basic dynamical features of a self-map $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ (pseudoconvexity and hyperbolicity of the Fatou set, connectedness of the Julia set). We prove the following theorem.

Theorem 1. *Let $\mathbb{P}E \to \mathbb{P}^n$ be a projective bundle with nonzero discriminant, and let $\mathbb{P}E \xrightarrow{f} \mathbb{P}E$ be a self-map with topological degree at least* 2. *Then*:

(1) *f has well-defined algebraic degree*;
(2) *the Fatou components of f are Stein and Kobayashi hyperbolic*;
(3) *when* $\operatorname{rank}(E) \geq 3$, *the Julia set of f is connected.*

Note that the first two statements fail in the trivial case $\mathbb{P}E = \mathbb{P}^n \times \mathbb{P}^n$.

Preliminaries. Fix a vector bundle $E \xrightarrow{\pi'} B$, $\operatorname{rank}(E) = r + 1$, over a projective manifold B, $\dim(B) = n$, and let $E' \xrightarrow{\pi} B$ denote its dual. The homogeneous lines in E' form the projective manifold $\mathbb{P}E$, which is endowed with the projection $\mathbb{P}E \xrightarrow{p} B$. The pull-back p^*E' admits a canonical line sub-bundle,

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