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Horocyclically Convex Univalent Functions

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1. Introduction: Convexity and Curvature

A domain G in $\mathbb{C}$ is convex if and only if, for every $\omega \in \partial G$, there exists a half-plane H such that $\omega \in \partial H$ and $G \cap H = \emptyset$. The fact that $C = \partial H$ is a line can be expressed in two equivalent ways:

(i) C is a maximal curve of zero curvature;
(ii) C is a noncompact maximal curve of constant curvature.

Now we turn to the hyperbolic metric $ds = (1 - |z|^2)^{-1}|dz|$ in the unit disk $\mathbb{D}$. Let $C : w(s)$, $s \in I$, be a curve of class $\mathcal{C}^2$ in $\mathbb{D}$ that is parameterized by the hyperbolic arc-length s; that is,

$$|w'| = 1 - |w|^2, \quad w'' \text{ is continuous in } I. \tag{1.1}$$

Then the hyperbolic (= geodesic) curvature κ satisfies the differential equation

$$\frac{w''}{w'} + \frac{2\bar{w}w'}{1 - |w|^2} = i\kappa. \tag{1.2}$$

Let $\mathbb{T} = \partial\mathbb{D}$. If $C \subset \mathbb{D}$ is a maximal (= cannot be extended to a larger curve in $\mathbb{D}$) curve of constant hyperbolic curvature κ, then we have three cases as follows.

$|\kappa| \le 2$: C is a circular arc from $\mathbb{T}$ to $\mathbb{T}$;
$|\kappa| = 2$: C is a circle in $\mathbb{D}$ that touches $\mathbb{T}$;
$|\kappa| > 2$: C is a full circle in $\mathbb{D}$.

Thus the noncompact maximal curves of constant curvature are those with $|\kappa| \le 2$, and we see that conditions (i) and (ii) are different in the hyperbolic case.

Ma and Minda [MaM1] considered condition (i). A domain $G \subset \mathbb{D}$ is hyperbolically convex (h-convex) if, for every $\omega \in \mathbb{D} \cap \partial G$, there is a hyperbolic half-plane H with $\omega \in \partial H$ and $G \cap H = \emptyset$. See, for example, [MaM2; MeP1; MeP2; MeP3; MePV] for results on the conformal maps of $\mathbb{D}$ onto h-convex domains.

In this paper we shall consider condition (ii), namely, the extremal case $|\kappa| = 2$. A *horocycle* is, by definition, the inner domain of a circle in $\mathbb{D}$ that touches $\mathbb{T}$. A domain $G \subset \mathbb{D}$ will be called *horocyclically convex* (horo-convex) if, for every $\omega \in \mathbb{D} \cap \partial G$, there exists a horocycle H such that

$$\omega \in \partial H \quad \text{and} \quad G \cap H = \emptyset. \tag{1.3}$$

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