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Isospectral Metrics and Potentials on Classical Compact Simple Lie Groups

Emily Proctor

1. Introduction

Given a compact Riemannian manifold (M, g), the eigenvalues of the Laplace operator Δ form a discrete sequence known as the *spectrum* of (M, g). (In the case of M with boundary, we stipulate either Dirichlet or Neumann boundary conditions.) We say that two Riemannian manifolds are *isospectral* if they have the same spectrum. For a fixed manifold M, an isospectral deformation of a metric g_0 on M is a continuous family $\mathcal{F}$ of metrics on M containing g_0 such that each metric $g \in \mathcal{F}$ is isospectral to g_0. We say that the deformation is *nontrivial* if none of the other metrics in $\mathcal{F}$ are isometric to g_0 and that the deformation is *multidimensional* if $\mathcal{F}$ can be parameterized by more than one variable. For two functions $\phi, \psi \in C^\infty(M)$, we say that ϕ and ψ are isospectral *potentials* on (M, g) if the eigenvalue spectra of the Schrödinger operators $\hbar^2\Delta + \phi$ and $\hbar^2\Delta + \psi$ are equal for any choice of Planck's constant $\hbar$.

In this paper, we prove the existence of multiparameter isospectral deformations of metrics on $\mathrm{SO}(n)$ ($n = 9$ or $n \geq 11$), $\mathrm{SU}(n)$ ($n \geq 8$), and $\mathrm{Sp}(n)$ ($n \geq 4$). For these examples we follow a metric construction developed by Schueth, who had given one-parameter families of isospectral metrics on orthogonal and unitary groups. Our multiparameter families are obtained by a new proof of nontriviality that establishes a generic condition for nonisometry of metrics arising from the construction. We also show the existence of noncongruent pairs of isospectral potentials and nonisometric pairs of isospectral conformally equivalent metrics on $\mathrm{Sp}(n)$ for $n \geq 6$.

The industry of producing isospectral manifolds began in 1964 with Milnor's pair of 16-dimensional isospectral, nonisometric tori [M]. Several years later, in the early 1980s, new examples began to appear sporadically (e.g. [GW1; I; V]). These isospectral constructions were ad hoc and did not appear to be related until 1985, when Sunada began developing the first unified approach for producing isospectral manifolds. The method described a program for taking quotients of a given manifold so that the resulting manifolds were isospectral. Sunada's original theorem and subsequent generalizations [Be1; Be2; DG; P; Su] explained most of the previously known isospectral examples and led to a wide variety of new ones; see, for example, [BGG], [Bu], and [GWW].

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