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Linearity of Sets of Strange Functions

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1. Introduction

In analysis, sometimes very strange phenomena appear. For instance, one should mention continuous nowhere differentiable functions, everywhere divergent Fourier series of functions in $L^1(\mathbb{T})$, or universal Taylor series. By experience, it is known that as soon as such a pathological example is exhibited, it is most often generic in the sense of Baire's categories. Namely, in a well-chosen topological space, all elements of a dense G_δ set share this pathological behavior.

More recently, the algebraic structure of these sets has been investigated (see e.g. [Ro] or [AGM]). Let us recall the following definition (introduced in [GuQ]).

Definition 1. A set M in a linear topological space X is said to be *spaceable* if $M \cup \{0\}$ contains a closed infinite-dimensional subspace of X.

In this paper, we give several examples of sets of functions with irregular behavior that are spaceable. Our main tool is the use of basic sequences, a technique initiated in this context by Bernal-Gonzalez and Montes-Rodriguez [BeMo; Mo] in the particular case of hypercyclic vectors. We recall some basic definitions and results, which are taken from [Di]. A sequence $(x_n)_{n\geq 1}$ of a Banach space X is called a *basic sequence* if, for each x belonging to $X_0 = \overline{\operatorname{span}}(x_n : n \geq 1)$, there exists a unique sequence of scalars (α_n) such that $x = \sum_{n=1}^{+\infty} \alpha_n x_n$. The coefficient functionals are defined by $x_k^*\bigl(\sum_{n=1}^{+\infty} \alpha_n x_n\bigr) = \alpha_k$. They are continuous on X_0 and can be extended to X by the Hahn–Banach theorem. Two basic sequences (x_n) and (y_n) are *equivalent* if the convergence of $\sum \alpha_n x_n$ is equivalent to the convergence of $\sum \alpha_n y_n$. We will intensively use the following result (see [Di, Thm. 9]).

Lemma 1. *Let (x_n) be a basic sequence in X, and let (y_n) be a sequence in X satisfying $\sum \|x_n^*\| \|x_n - y_n\| < 1$. Then (y_n) is a basic sequence equivalent to (x_n).*

This lemma explains our strategy for building large subspaces of functions with strange behavior. First, we exhibit in the space a basic sequence of functions with a very regular behavior. Next, we slightly disturb these functions, so that the new functions behave very irregularly, yet with the new sequence remaining a basic sequence. Finally, we show that a good choice of the perturbations ensures that the irregular behavior transfers to the subspace generated by the basic sequence. This

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