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Group Actions on Stacks and Applications

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The motivation at the origin of this article is to investigate some ways of constructing moduli for curves, and covers above them, using tools from stack theory. This idea arose from reading Bertin–Mézard [BeM, esp. Sec. 5] and Abramovich–Corti–Vistoli [ACV]. Our approach is in the spirit of most recent works, where one uses the flexibility of the language of algebraic stacks. This language has two (twin) aspects: category-theoretic on one side and geometric on the other. Some of our arguments, especially in Section 8, are formal arguments involving general constructions concerning group actions on algebraic stacks (this is more on the categoric side). They are, intrinsically, natural enough to preserve the "modular" aspect. In trying to isolate these arguments, we were led to write results of independent interest. It seemed therefore more adequate to present them in a separate, self-contained part. Thus the article is split into two parts of comparable size.

More specifically, groups are ubiquitous in algebraic geometry (when one focuses on curves and maps between them, examples include the automorphism group, fundamental group, monodromy group, permutation group of the ramification points, ...). It is natural to ask whether we can handle group actions on stacks in the same fashion as we do on schemes. For example, we expect: that the quotient of the stack of curves with ordered marked points $\mathcal{M}_{g,n}$ by the symmetric group should classify curves with unordered marked points; that if G acts on a scheme X then the fixed points of the stack $\mathcal{P}ic(X)$ under G should be related to G-linearized line bundles on X; and that the quotient of the modular stack curve $\mathcal{X}_1(N)$ by $(\mathbb{Z}/N\mathbb{Z})^\times$ should be $\mathcal{X}_0(N)$ (the notation is, we hope, well known to the reader). Other important examples appear in the literature: action of tori on stacks of stable maps in Gromov–Witten theory [Ko; GrPa], and action of the symmetric group $\mathfrak{S}_d$ on a stack of multisections in [L-MB, (6.6)]. Our aim is to provide the material necessary to handle the questions raised here and then answer them, as well as to give other applications.

Let us now explain in more detail the structure and results of this paper.

In Part A we discuss the notion of a group action on a stack. We are mainly interested in giving general conditions under which the fixed points and the quotient of an algebraic stack are algebraic. In Sections 1 and 2 we give definitions and basics on actions. For simplicity let us now consider a flat group scheme G and an algebraic stack $\mathcal{M}$, both of finite presentation (abbreviated fp) over some

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