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A Counterexample to the Fourteenth Problem of Hilbert in Dimension Three

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1. Introduction

Let K be a field, $K[X] = K[X_1, \dots, X_m]$ the polynomial ring in m variables over K, and $K(X)$ its field of fractions. Then, the fourteenth problem of Hilbert asks whether the K-subalgebra $L \cap K[X]$ of $K[X]$ is finitely generated whenever L is a subfield of $K(X)$ containing K. Zariski [23] showed that $L \cap K[X]$ is finitely generated if the transcendence degree of L over K is at most two. Consequently, the problem has an affirmative answer if $m \le 2$. On the other hand, a counterexample to the problem was first found by Nagata [17] in 1958 for $m \ge 32$ (see [8] for the progress on this problem).

Recently the author [12] gave a counterexample for $m = 4$, whereby the problem remained open only for $m = 3$. In fact, if $L \cap K[X]$ is not finitely generated, then $L \cap K[X][X_{m+1}, \dots, X_{m+r}]$ is also not finitely generated for each $r \ge 0$. In this paper we give the first counterexample to the problem for $m = 3$. Thus, the fourteenth problem of Hilbert is settled for all m at last.

Let γ and $\delta_{i,j}$ be positive integers for $i, j = 1, 2$, and let

$$\begin{aligned} \Pi_1 &= X_1^{\delta_{2,1}} X_2^{-\delta_{2,2}} - X_1^{-\delta_{1,1}} X_2^{\delta_{1,2}}, \\ \Pi_2 &= X_3^{\gamma} - X_1^{-\delta_{1,1}} X_2^{\delta_{1,2}}, \\ \Pi_3 &= 2 X_1^{\delta_{2,1} - \delta_{1,1}} X_2^{\delta_{1,2} - \delta_{2,2}} - X_1^{-2\delta_{1,1}} X_2^{2\delta_{1,2}}. \end{aligned} \tag{1.1}$$

Then we have the following result.

Theorem 1.1. *Assume that the characteristic of K is zero. If*

$$\frac{\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}} + \frac{\delta_{2,2}}{\delta_{2,2} + \delta_{1,2}} < \frac{1}{2}, \tag{1.2}$$

then $K(\Pi_1, \Pi_2, \Pi_3) \cap K[X_1, X_2, X_3]$ is not finitely generated over K.

We remark that $m = 3$ is an exceptional dimension for the fourteenth problem of Hilbert with many partial positive answers as follows. As already mentioned, the answer to the fourteenth problem of Hilbert is affirmative when $m \le 2$ by

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