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On the Semisimplicity of Cyclotomic Temperley–Lieb Algebras

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1. Introduction

The Temperley–Lieb algebras were first introduced in [15] in order to study the single-bond transfer matrices for the Ising model and for the Potts model. Jones [9] defined a trace function on a Temperley–Lieb algebra so that he could construct the Jones polynomial of a link when the trace is nondegenerate. It is known that the trace is nondegenerate if the Temperley–Lieb algebra is semisimple. So it is an interesting question to provide a criterion for a Temperley–Lieb algebra to be semisimple. In [16, Sec. 5], there is a simple criterion for the semisimplicity of the Temperley–Lieb algebra in terms of q if the parameter is written $\delta = -(q+q^{-1})$. More explicitly, Westbury computed the determinants of Gram matrices associated to all "cell modules" via Tchebychev polynomials. This implies that a Temperley–Lieb algebra is semisimple if and only if such polynomials do not take values zero for the parameters.

As a generalization of a Temperley–Lieb algebra, the cyclotomic Temperley–Lieb algebra $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ of type $G(m,1,n)$ was introduced in [13]. It is proved in [13] that $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is a cellular algebra in the sense of [3]. Thus $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is semisimple if and only if all of its "cell modules" are pairwise nonisomorphic irreducible. In order to determine when a cell module is irreducible, Rui and Xi computed the determinants of Gram matrices of certain cell modules [13, 8.1]. In general, it is hard to compute the determinants for all cell modules.

In this note, we shall consider the semisimplicity of cyclotomic Temperley–Lieb algebras. This is analogous to the question considered in [14] (see [2] for the case $m = 1$). Following [11], we study two functors F and G between certain categories in Section 3. Via these functors and [13, 8.1], in Section 4 we show our main result (Theorem 4.6), which states that $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is semisimple if and only if generalized Tchebychev polynomials do not take values zero for the parameters $\bar{\delta}_i$, $1 \le i \le m$.

2. Cyclotomic Temperley–Lieb Algebras

In this section, we recall some of results on the cyclotomic Temperley–Lieb algebras in [13]. Throughout the paper, we fix two natural numbers m and n.

A labeled Temperley–Lieb diagram (or labeled TL diagram) D of type $G(m,1,n)$ is a Temperley–Lieb diagram with $2n$ vertices and n arcs. Each arc is labeled by

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