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Initial Algebras of Determinantal Rings, Cohen–Macaulay and Ulrich Ideals

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1. Introduction

Let K be a field and X an $m \times n$ matrix of indeterminates over K. Let $K[X]$ denote the polynomial ring generated by all the indeterminates X_{ij}. For a given positive integer $r \le \min\{m,n\}$, we consider the determinantal ideal $I_{r+1} = I_{r+1}(X)$ generated by all $r+1$ minors of X if $r < \min\{m,n\}$ and $I_{r+1} = (0)$ otherwise. Let $R_{r+1} = R_{r+1}(X)$ be the determinantal ring $K[X]/I_{r+1}$.

Determinantal ideals and rings are well-known objects, and the study of these objects has many connections with algebraic geometry, invariant theory, representation theory, and combinatorics. See Bruns and Vetter [BrV] for a detailed discussion.

In the first part of this paper we develop an approach to determinantal rings via initial algebras. We cannot prove new structural results on the rings R_{r+1} in this way, but the combinatorial arguments involved are extremely simple. They yield quickly that R_{r+1}, with respect to its classical generic point, has a normal semigroup algebra as its initial algebra. Using general results about toric deformations and the properties of normal semigroup rings, one obtains immediately that R_{r+1} is normal and Cohen–Macaulay, has rational singularities in characteristic 0, and is F-rational in characteristic p.

Toric deformations of determinantal rings have been constructed by Sturmfels [St] for the coordinate rings of Grassmannians (via initial algebras) and by Gonciulea and Lakshmibai [GoL] for the class of rings considered by us. The advantage of our approach, compared to that of [GoL], is its simplicity.

Moreover, it allows us to determine the Cohen–Macaulay and Ulrich ideals of R_{r+1}. Suppose that $1 \le r < \min\{m,n\}$ and let $\mathfrak{p}$ (resp., $\mathfrak{q}$) be the ideal of R_{r+1} generated by the r-minors of the first r rows (resp. the first r columns) of the matrix X. The ideals $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of height 1 and hence they are divisorial, because R_{r+1} is a normal domain. The divisor class group $\mathrm{Cl}(R_{r+1})$ is isomorphic to $\mathbb{Z}$ and is generated by the class $[\mathfrak{p}] = -[\mathfrak{q}]$ (see [BrH, Sec. 7.3; BrV, Sec. 8]). The symbolic powers of $\mathfrak{p}$ and $\mathfrak{q}$ coincide with the ordinary ones. Therefore, the ideals $\mathfrak{p}^k$ and $\mathfrak{q}^k$ represent all reflexive rank-1 modules. The goal of Section 4 is to show that $\mathfrak{p}^k$ (resp., $\mathfrak{q}^k$) is a Cohen–Macaulay ideal if and only if $k \le m - r$ (resp., $k \le n - r$). In addition, we prove that the powers $\mathfrak{p}^{m-r}$ and $\mathfrak{q}^{n-r}$ are even Ulrich ideals.

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