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# Automorphisms of Affine Surfaces with $\mathbb{A}^1$-Fibrations

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## 1. Introduction

Let $X$ be a normal affine surface defined over the complex field $\mathbf{C}$, which has at worst quotient singularities. We call $X$ simply a *log affine surface*. If further $H_i(X;\mathbb{Q}) = (0)$ for $i > 0$ then $X$ is called a *log $\mathbb{Q}$-homology plane* (if $X$ is smooth then $X$ is simply called a $\mathbb{Q}$-homology plane). Let $G_a$ denote the complex numbers with addition as an algebraic group. In this paper we are mainly interested in log affine surfaces $X$ that have an $\mathbb{A}^1$-fibration. Of particular interest are surfaces that admit a regular action of $G_a$. Such actions up to conjugacy correspond in a bijective manner to $\mathbb{A}^1$-fibrations on $X$ with base a smooth affine curve. Algebraically, these actions correspond bijectively to locally nilpotent derivations of the coordinate ring $\Gamma(X)$ of $X$. The set of all elements of $\Gamma(X)$ that are killed under all the locally nilpotent derivations of $\Gamma(X)$ is called the *Makar-Limanov invariant* of $X$ and denoted by $\mathrm{ML}(X)$.

If a smooth affine surface has two independent $G_a$ actions then its Makar-Limanov invariant is trivial. Gizatullin [9] and Bertin [2] gave a necessary and sufficient condition for this to happen. More recently, Bandman and Makar-Limanov [1] proved that a smooth affine surface $X$ with trivial canonical bundle and $\mathrm{ML}(X) = \mathbf{C}$ is an affine surface in $\mathbb{A}^3$ defined by $\{xy = p(z)\}$, where $p(z)$ is a polynomial with distinct roots. Masuda and Miyanishi [12] applied this to determine the structure of a $\mathbb{Q}$-homology plane with trivial ML-invariant. They proved that such a surface is a quotient of the Bandman–Makar-Limanov hypersurface by the action of a finite cyclic group (see result (3) in the listing that follows).

In this paper we extend the last result to the case of log $\mathbb{Q}$-homology planes in Section 2. Similar and related results in Section 2 and Section 3 have been obtained independently by Daigle and Russell [4] and Dubouloz [5]. An automorphism of a smooth affine surface sends fibers of one $\mathbb{A}^1$-fibration with affine base to the fibers of another $\mathbb{A}^1$-fibration. If these two fibrations are different then the Makar-Limanov invariant of the surface is trivial. If a smooth affine surface has an $\mathbb{A}^1$-fibration whose base is not an affine curve, then this fibration does not correspond to a $G_a$ action. In this case the geometry of the fibration enters into the picture. In Section 4 we give a sufficient condition for uniqueness of an $\mathbb{A}^1$-fibration on a smooth affine surface. This involves the number of multiple fibers

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