Introduction to the Toric Mori Theory

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1. Introduction

The main purpose of this paper is to give a simple and noncombinatorial proof of the toric Mori theory. Here, the toric Mori theory means the (log) minimal model program (MMP, for short) for toric varieties.

In his famous and beautiful paper [R], Reid carried out the toric Mori theory under the assumption that the variety is *complete*. His arguments are combinatorial. Thus, it is not so obvious whether we can remove the completeness assumption from his paper. We quote his idea from [R].

> (0.3) *Remarks.* The hypothesis that A is complete is not essential; it can be reduced to the projective case, or possibly eliminated by a careful (and rather tedious) rephrasing of the arguments of §§1–3. The projectivity hypothesis on f is needed in order for the statement of (0.2) to make sense, since without projectivity the cone NE(V/A) will usually not have any extremal rays.

We prefer not to simply rephrase his approach, which entails tedious combinatorial arguments. Instead, our proof (which is independent of Reid's proof) heavily relies on the general machinery of the MMP and the special properties of toric varieties. Thus, our proof works without the completeness assumption.

For the details of the toric Mori theory, see [O, Sec. 2.5; KMM, Sec. 5.2; OP; L; I2; Ma, Ch. 14; W; Fj1]. Matsuki [Ma] corrected some minor errors in [R] and pointed out some ambiguities in [R] and [KMM]; see Remarks 14-1-3(ii), 14-2-3, and 14-2-7 in [Ma]. We believe that these remarks help the reader to understand [R]. We recommend that the reader compare this paper with [Ma, Ch. 14]. Shokurov treats the MMP for toric varieties in a noncombinatorial way (see [Sh, Ex. 3]). His arguments are quite different from ours. For the more advanced topics of toric Mori theory, see [Fj2]. For the outline of the general MMP, which is still conjectural in dimension ≥ 4, see [KMM, Introduction] or [KoMo, 2.14, Sec. 3.7].

We note that the Zariski decomposition on toric varieties has already been treated by various researchers. The reason we treat it here is to show that the Zariski