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Dynamics of Quadratic Polynomial Mappings of $\mathbb{C}^2$

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Introduction

A remarkable feature of one-dimensional complex dynamics is the prominent role played by the "quadratic family" $P_c(z) = z^2 + c$. The latter has revealed an exciting source of study and inspiration for the study of general rational mappings $f: \mathbb{P}^1 \to \mathbb{P}^1$ as well as for more general dynamical systems [L]. Our purpose here is to introduce several quadratic families of polynomial self-mappings of $\mathbb{C}^2$ that we hope will be the complex two-dimensional counterpart to the celebrated quadratic family.

We partially classify quadratic polynomial endomorphisms of $\mathbb{C}^2$ (see Section 2) using some numerical invariants (dynamical degrees $\lambda_1(f), d_t(f)$ and dynamical Lojasiewicz exponent $DL_\infty(f)$), which we define in Section 1. We then use this classification to test two related questions.

Question 1. Does there exist a unique invariant probability measure of maximal entropy?

Question 2. Does there exist an algebraically stable compactification?

Simple examples show that there may be infinitely many invariant probability measures of maximal entropy when $d_t(f) = \lambda_1(f)$. When $d_t(f) > \lambda_1(f)$, it is proved in [Gu2] that the Russakovskii–Shiffman measure μ_f is the unique measure of maximal entropy. We push further the study of μ_f, when f is quadratic, by showing that it is compactly supported in $\mathbb{C}^2$ (Section 4). Moreover, every plurisubharmonic function is in $L^1(\mu_f)$ (Section 5) and the "exceptional set" is algebraic (Section 6).

When $d_t(f) < \lambda_1(f)$, one also expects the existence of a unique measure of maximal entropy (this is the case when f is a complex Hénon mapping [BLS1]). If f is algebraically stable on some smooth compactification of $\mathbb{C}^2$, then one can construct invariant currents T_+, T_- such that $f^*T_+ = \lambda_1(f)T_+$ and $f_*T_- = \lambda_1(f)T_-$ (see [Gu1]). It is usually difficult to define the invariant measure $\mu_f = T_+ \wedge T_-$. However, this can be done when f is polynomial in $\mathbb{C}^2$, since T_+ admits continuous potentials off a finite set of points. We briefly discuss Question 1 for quadratic mappings with $d_t(f) < \lambda_1(f)$ in Section 3; the answer is positive for an open set of parameters but unknown in general.

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