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$\mathbf{C}_+$-Actions on Contractible Threefolds

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1. Introduction

The aim of this paper is to generalize the theorem of Miyanishi [M1] stating that, for any nontrivial algebraic $\mathbf{C}_+$-action on $\mathbf{C}^3$, the algebraic quotient $\mathbf{C}^3/\!/\mathbf{C}_+$ is isomorphic to $\mathbf{C}^2$. Our main result is that, for a nontrivial algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X, the algebraic quotient $X/\!/\mathbf{C}_+$ is isomorphic to a smooth contractible affine surface S. Since all such surfaces are rational [GS], we deduce that X is rational as well. Furthermore, if the action is free, then we conclude that X is isomorphic to $S \times \mathbf{C}$ and that the action is induced by translation on the second factor by virtue of [K3], where this result was proved under the additional assumption that S is smooth. Another consequence of our main result is that, when X admits a dominant morphism from a threefold of form $C \times \mathbf{C}^2$, the quotient S is isomorphic to $\mathbf{C}^2$. We also give an independent proof of the latter fact that (unlike our main result) does not use the difficult theorem of Taubes [T] about the absence of simply connected homology cobordisms between certain homology spheres. In fact, the rationality of X can also be proved without this theorem; however, this would require another difficult theorem that all logarithmic $\mathbf{Q}$-homology planes are rational [PS; GPS; GP]. In conclusion, we derive the following criterion: If there is a free algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X that admits a dominant morphism from $C \times \mathbf{C}^2$, then X is isomorphic to $\mathbf{C}^3$.

2. The Main Result

Let $\rho: X \to S$ be the quotient morphism of a nontrivial algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X. By Fujita's result, X is factorial (see e.g. [K1]). Some other properties of $\rho: X \to S$ proved in [K3, Lemma 2.1, Prop. 3.2, Rem. 3.3] are summarized in the following lemma.

Lemma 2.1.

(1) *The surface S is affine and factorial, and $\rho^{-1}(s)$ is a nonempty curve for every $s \in S$.*

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