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Projective Geometry of Freudenthal's Varieties of Certain Type

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Dedicated to Professor Hiroshi Asano on the occasion of his 70th birthday

0. Introduction

H. Freudenthal constructed, in a series of his papers (see [10] and its references), the exceptional Lie algebras of type E_8, E_7, E_6, and F_4 by defining various projective varieties. The purpose of our work is to study projective geometry for his varieties of certain type, which are called *varieties of planes* in the symplectic geometry of Freudenthal (see [10, 4.11] and [23, 2.3]).

Let $\mathfrak{g}$ be a graded, simple, finite-dimensional Lie algebra over the complex number field $\mathbb{C}$ with grades between -2 and 2, $\dim \mathfrak{g}_2 = 1$, and $\mathfrak{g}_1 \neq 0$, namely, a *graded Lie algebra of contact type*: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see Section 1). We set

$$\mathcal{V} := \{x \in \mathfrak{g}_1 \setminus \{0\} \mid (\operatorname{ad} x)^2 \mathfrak{g}_{-2} = 0\},$$

and define an algebraic set V in $\mathbb{P}(\mathfrak{g}_1)$ to be the projectivization of $\mathcal{V}$:

$$V := \pi(\mathcal{V}),$$

where $\pi : \mathfrak{g}_1 \setminus \{0\} \to \mathbb{P}(\mathfrak{g}_1)$ is the natural projection. Then we call $V \subseteq \mathbb{P}(\mathfrak{g}_1)$ (with the reduced structure) the *Freudenthal variety* associated to the graded Lie algebra $\mathfrak{g}$ of contact type, which is a natural generalization of Freudenthal's varieties mentioned previously. Note that V is not necessarily connected in this general setting. Moreover, we here consider the projectivization of a closed set $\{x \in \mathfrak{g}_1 \mid (\operatorname{ad} x)^{k+1} \mathfrak{g}_{-2} = 0\}$ and denote it by V_k; we have

$$\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \mathbb{P},$$

where we set $\mathbb{P} := \mathbb{P}(\mathfrak{g}_1)$ for short. Clearly, V_3 is a quartic hypersurface, V_2 is an intersection of cubics, and $V_1 = V$ is an intersection of quadrics (with a few exceptions).

In the literature, several results have been known about the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-space—case by case for each exceptional Lie algebra of types E_8, E_7, E_6, and F_4—from the viewpoint of the invariant theory of prehomogeneous vector spaces (see [13; 15; 19; 22]). By virtue of those results, it can be shown, for example, that the stratification of $\mathbb{P}$ given by the differences of the V_k exactly corresponds to the

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