Michigan Math. J. 52 (2004)

Vertex Operator Realizations of Conformal Superalgebras

Michael Roitman

Introduction

One of the first origins of vertex algebras was the explicit constructions of representations of certain Lie algebras by means of so-called vertex operators. The first construction of this kind was done by Lepowsky and Wilson [22], who constructed a vertex operator representation of the affine algebra $A_1^{(1)}$. Their work was later generalized in [17]. Frenkel and Kac [10] and, independently, Segal [30] constructed the basic representations of the simply laced affine Lie algebras using *untwisted vertex operators* (as opposed to the twisted vertex operators of Lepowsky and Wilson). Later vertex operators were used to construct a large family of modules for different types of Lie algebras, including all of the affine Kac–Moody algebras, toroidal algebras, and some other extended affine Lie algebras (see e.g. [3; 11; 12; 13; 25; 32] and references therein). The advantage of vertex operator constructions is that they are very explicit. They have yielded many interesting results for combinatorial identities, modular forms, soliton theory, and so forth.

It seems to be a natural problem to describe all Lie algebras, or at least a large family of Lie algebras, for which the vertex operator constructions of representations work. Our first observation is that, in some of the cases just described, the Lie algebras whose representations are constructed by vertex operators correspond to the *conformal algebras* introduced by Kac [15; 16]; see also [26; 27; 33]. On the other hand, vertex operators give rise to another algebraic structure, called *vertex algebras,* studied extensively in [4; 9; 11; 15], for example. A vertex operator construction of representations of Lie algebras amounts sometimes to an embedding of a conformal algebra into a vertex algebra generated by vertex operators, so that the vertex algebra becomes an *enveloping vertex algebra* of these conformal algebras.

In the present work we make the first step in describing the Lie algebras that are representable by vertex operators. We classify the Lie algebras that can be realized by the untwisted vertex operators of Frenkel–Kac–Segal. The vertex algebra generated by these vertex operators is also called *lattice vertex algebra* because its construction depends on a choice of an integer lattice. In fact there is a construction that, for every lattice Λ with an integer-valued bilinear form $(\cdot|\cdot)$, gives

Received March 4, 2003. Revision received Decembr 8, 2003.
This research was partially conducted by the author for the Clay Mathematics Institute.