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Blow-up of Positive Solutions for a Family of Nonlinear Parabolic Equations in General Domain in $\mathbb{R}^N$

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1. Introduction

We consider the following nonlinear parabolic problem,

$$u_t - \triangle u = F(u, \nabla u), \quad x \in \Omega,\ t > 0 \tag{1.1}$$

$$u(t,x) = 0, \quad x \in \partial\Omega, \tag{1.2}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where Ω is a bounded (or unbounded) and sufficiently regular (say, uniformly regular of class C^2) open domain in $\mathbb{R}^N$, $F \in C^1(\mathbb{R} \times \mathbb{R}^N)$, and u_0 satisfies the compatibility condition (i.e., $u_0(x) = 0$ on $\partial\Omega$). It is well known that the problem (1.1)–(1.3) admits a unique classical solution u, of maximal existence time $T^* \in (0,\infty]$, when Ω is a bounded domain and $u_0(x) \in C^1(\bar{\Omega})$ [14, Thm. 10, p. 206]. Moreover, if $T^* < \infty$, then u blows up in finite time in C^1 norm; that is,

$$\limsup_{t\to T^*} \sup_{x\in\bar{\Omega}} |u(x,t)| + |\nabla u(x,t)| = +\infty.$$

It is also known that if $F(u,\nabla u) = b|\nabla u|^p + au^q$ ($p > 1$, $q > 1$, and $a, b \in \mathbb{R}$) then (1.1)–(1.3) admits a unique, maximal-in-time solution $u \in C([0,T^*); W_0^{1,s}(\Omega))$ for all sufficiently regular initial data. For example, $u_0 \in W_0^{1,s}(\Omega)$ with $s \geq N\max(p,q)$ when Ω is an unbounded domain. Moreover, if $T^* < \infty$ then $\lim_{t\to T^*}\|u(t)\|_{W_0^{1,\infty}(\Omega)} = \infty$.

The foregoing two regularity assumptions on u_0 will be maintained throughout the paper.

The equation

$$u_t - \triangle u = |\nabla u|^p, \quad t > 0,\ x \in \Omega,\ p > 1, \tag{1.4}$$

serves as a typical model case in the theory of parabolic partial differential equations. In fact, it is the simplest example of parabolic PDE with a nonlinearity depending on the first-order spatial derivatives of u, and it can be considered as an analogue of the extensively studied equation with zero-order nonlinearity, $u_t - \triangle u = u|u|^{p-1}$. This equation was studied by many authors in the past few years [3; 4; 5; 6; 7; 8; 16; 17; 21; 22; 29].

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